Sec 3.5 Derivatives of Trig Functions

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cosh - 1)}{h} + \frac{\cos x \sin h}{h} = \lim_{h\to 0} \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \sin x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h}$$

$$= \lim_{h\to 0} \sin x \cdot 0 + \lim_{h\to 0} \cos x \cdot 1$$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

You try: $\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\cos x(\cosh - 1)}{h}} - \frac{\sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot 0 - \lim_{h \to 0} \sin x \cdot 1$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$$\boxed{\frac{d}{dx}(\cot x) = -\csc^2 x}$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right)$$

$$= \frac{\cos x \cdot 0 - 1(-\sin x)}{\cos^2 x} = \frac{1 \cdot \sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

$$\frac{d}{dx}(\csc x) = \frac{d}{dx}\left(\frac{1}{\sin x}\right)$$

$$= \frac{\sin x \cdot 0 - 1 \cdot \cos x}{\sin^2 x} = \frac{-1 \cos x}{\sin^2 x} = \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x}$$

$$\boxed{\frac{d}{dx}(\csc x) = -\csc x \cot x}$$

Examples

① $y = x^2 \sin x$

$$\boxed{y' = x^2 \cos x + 2x \sin x}$$

② $y = \dfrac{\sec x}{x}$ $\quad y' = \boxed{\dfrac{x \sec x \tan x - \sec x}{x^2}}$

③ $y = \dfrac{\cos x}{1 - \sin x}$ $\quad y' = \dfrac{(1 - \sin x)(-\sin x) - \cos x\,(0 - \cos x)}{(1 - \sin x)^2}$

$$y' = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} = \boxed{\frac{1}{1 - \sin x}}$$

4. $f(x) = \dfrac{\sec x}{1 + \tan x}$

$$f'(x) = \frac{(1 + \tan x)\sec x \tan x - \sec x(\sec^2 x)}{(1 + \tan x)^2}$$

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x\left[\tan x + \overbrace{\tan^2 x - \sec^2 x}^{-1}\right]}{(1 + \tan x)^2}$$

$$= \frac{\sec x\,[\tan x - 1]}{(1 + \tan x)^2}$$

$\sec^2 x - \tan^2 x = 1$