APPLICATIONS OF DERIVATIVES

AP Exam Review

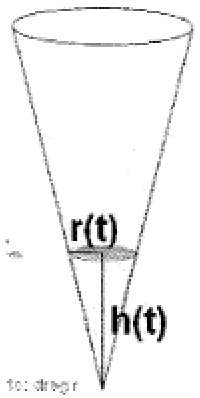
RELATED RATES

When working related rate problems, instead of finding a derivative of an equation y with respect to x, you are finding the derivative of equations with respect to a "hidden" variable t. The variables are tied together in their relationship to time.

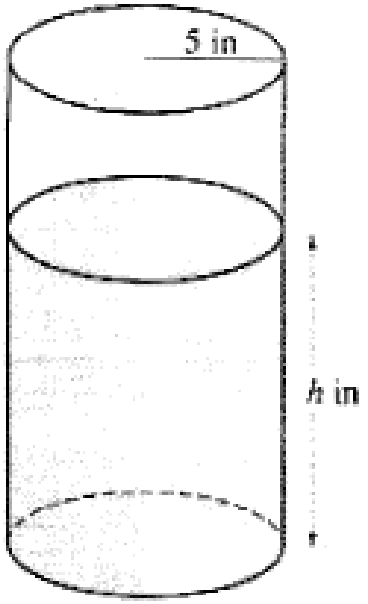
- Draw a picture to model the problem.
- Write an equation that reflects the model. (Pythagorean theorem, Trig, Similar Figures like cones.
- Plug in values that never change.
- Implicit Differentiation with respect to time.
- Plug in values you know, and solve for what you are looking for.

Typical Example: A 6 meter ladder is against the wall. If the bottom is pushed/pulled at a constant rate 0.5 m/sec, how fast is the ladder top sliding when it reaches 5 meters up the wall?

Typical Example #2: Water is flowing into a cone with height of 16 cm and radius of 4 cm at a rate of 2 cubic cm per minute. How fast is the water level rising when the water is 5 cm deep?



AP example.



5. A coffeepot has the shape of a cylinder with radius 5 inches, as shown in the figure above. Let h be the depth of the coffee in the pot, measured in inches, where h is a function of time t, measured in seconds. The volume V of coffee in the pot is changing at the rate of $-5\pi\sqrt{h}$ cubic inches per second. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)

(a) Show that $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$.

(b) Given that $h = 17$ at time $t = 0$, solve the differential equation $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$ for h as a function of t.

(c) At what time t is the coffeepot empty?

(a) $V = 25\pi h$

$$\frac{dV}{dt} = 25\pi\frac{dh}{dt} = -5\pi\sqrt{h}$$

$$\frac{dh}{dt} = \frac{-5\pi\sqrt{h}}{25\pi} = -\frac{\sqrt{h}}{5}$$

3 : { 1 : $\frac{dV}{dt} = -5\pi\sqrt{h}$
1 : computes $\frac{dV}{dt}$
1 : shows result }

(b) $\frac{dh}{dt} = -\frac{\sqrt{h}}{5}$

$$\frac{1}{\sqrt{h}}dh = -\frac{1}{5}dt$$

$$2\sqrt{h} = -\frac{1}{5}t + C$$

$$2\sqrt{17} = 0 + C$$

$$h = \left(-\frac{1}{10}t + \sqrt{17}\right)^2$$

5 : { 1 : separates variables
1 : antiderivatives
1 : constant of integration
1 : uses initial condition $h = 17$ when $t = 0$
1 : solves for h }

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

(c) $\left(-\frac{1}{10}t + \sqrt{17}\right)^2 = 0$

$$t = 10\sqrt{17}$$

1 : answer

LINEARIZATION

A linearization is just another term for tangent line, solved for y.

You can use a linearization to estimate the value of a function near the point of tangency used to create it.

SPEED, VELOCITY AND ACCELERATION

- The rate of change of the position function is velocity.
- The rate of change of the velocity function is acceleration.
- Velocity denotes direction as well as speed.
- Speed is the absolute value of velocity.
- If velocity and acceleration have the same sign, speed is increasing.
- If velocity and acceleration are opposite in sign, speed is decreasing.
- Understand the difference between average rate of change over an interval and instantaneous rate of change at a point.
- Understand the difference between distance traveled and displacement (distance from starting point).

CURVE SKETCHING

- Find maxima and minima.
  - First derivative test: Determine the critical numbers where f' is 0 or undefined, then make a sign chart to look for a change in direction.
  - Second derivative test: f" at critical # is positive, min. If f" at critical # is negative, max.
  - On closed interval, be sure to explicity check endpoints.
- Find points of inflection.
  - f" is zero or undefined, and there is a sign change.
- Find intervals of increasing/decreasing and concave up/down.
- Use the above characteristics to see relationship between graphs of f, f' and f".

Sign charts

- Sign charts are valuable tools and are allowed,
- BUT THEY ARE NEVER NEVER NEVER SUFFICIENT TO EARN A POINT
- To earn all test points you must interpret the sign chart using words
- Your words should demonstrate that you understand the connection between the positive/negative behavior of a graph and the increasing/decreasing/extrema behavior of the parent function and how the increasing/decreasing behavior determines the extrema behavior

Example: f(x) is increasing on x=[0,2] because f'(x) is positive.

Example: f(x) has a relative max at x = 2 because f'(x) is 0 at x=2 and f'(x) is positive for x<2 and negative for x>2 meaning f(x) is increasing then decreasing, respectively.

KEY THEOREMS

- Extreme Value Theorem: If f is continuous on a closed interval [a,b], then f has both a min and max on the interval.
- Mean Value Theorem: If f is continuous on a closed interval [a,b] and differentiable on the open interval (a,b), there exists a number c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. The slope of the tangent equals the slope of the secant.
- Rolle's Theorem: If f is continuous on a closed interval [a,b], differentiable on the open interval (a,b), and f(a) = f(b), then there exists a number c between a and b such that f'(c) = 0. There is a horizontal tangent at c. This is a special case of the Mean Value Theorem.

f is continuous on $[0, 10]$. What is the minimum number of horizontal tangents?

Solution: At least 2.

Justify.

One in $(2, 6)$. One in $(4, 10)$.

			c		d	
x	0	2	4	6	8	10
y	4	5	8	3	2	7
			5		5	

Horizontal tangents exist where $f'(x) = 0$.

By the Intermediate Value Theorem (IVT) there exists an x-value, c, in $(4, 6)$ where $f(c) = 5$.

By the same reasoning there exists an x-value, d, in $(8, 10)$ where $f(d) = 5$.

The average slope over $(2, c)$ is $\dfrac{f(c) - f(2)}{c - 2} = \dfrac{5 - 5}{c - 2} = 0$.

By the Mean Value Theorem for Derivatives (MVTD) there also exists an x-value in $(2, c)$ where $f' = 0$. Therefore, at least one horizontal tangent exists in $x = (2, 6)$.

By the same reasoning another horizontal tangent exists in the interval $(4, 10)$.

OPTIMIZATION

- Optimization problems are those that require you to determine such things as the greatest profit, least cost, minimum distance, greatest volume, etc.
  - 1. Set up an equation to maximize or minimize.
  - 2. Use substitution to get equation in one variable.
  - 3. Take the derivative.
  - 4. Set derivative equal to zero.
  - 5. Verify it is a max or min.
  - 6. Answer the question asked.

EXAMPLE

A rectangle has its base on the x-axis and its two upper corners on the parabola $y = 12 - x^2$. What is the largest possible area of the rectangle?