

No Calculator

1. Each of the limits is $f'(a)$ for some function f and some number "a"Identify f and "a" for each

$$\begin{aligned} f(x) &= 2^x \quad a=3 \\ f'(3) &= 2^3 \ln 2 \end{aligned}$$

$$\begin{aligned} f(x) &= 2 \sin x \quad a=3 \\ f'(3) &= 2 \cos 3 \end{aligned}$$

$$\begin{aligned} f(x) &= \sqrt{x} \quad a=1 \\ f'(1) &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} f(x) &= \tan x \quad a=\frac{\pi}{4} \\ f'(\frac{\pi}{4}) &= 2 \end{aligned}$$

$$\begin{aligned} f(x) &= x^2 \quad a=2 \\ f'(2) &= 4 \end{aligned}$$

$$\begin{aligned} f(x) &= \ln x \quad a=1 \\ f'(1) &= 1 \end{aligned}$$

2. Evaluate the following integrals:

(a) $\int \frac{x dx}{\sqrt{x^2 - 4}}$

$= x^2 - 4$

$du = 2x dx$

$\frac{1}{2} \int u^{-\frac{1}{2}} du$
 $u^{\frac{1}{2}}$

$\sqrt{x^2 - 4} + C$

(b) $\int \sin^{-1} x dx$

$$\begin{array}{c|c} D & S \\ \hline \sin^{-1} x & 1 \\ \hline & \sqrt{1-x^2} \end{array}$$

$$\begin{aligned} x \sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}} \\ u = 1-x^2 \\ du = -2x dx \\ x \sin^{-1} x + \sqrt{1-x^2} + C \end{aligned}$$

(c) $\int \frac{e^x}{x^2} dx$

$u = \frac{3}{x} = 3x^{-1}$

$du = -3x^{-2} dx$
 $= -\frac{3}{x^2} dx$

$-\frac{1}{3} \int e^u du =$
 $-\frac{1}{3} e^{\frac{3}{x}} + C$

(d) $\int \frac{x dx}{x^2 + 1}$

$u = x^2 + 1$

$du = 2x dx$

$\frac{1}{2} \ln |x^2 + 1| + C$

$\frac{1}{2} \ln |(x^2 + 1)| + C$

or
 $\frac{1}{2} \ln \sqrt{x^2 + 1} + C$

(e) $\int \frac{x dx}{\sqrt{1-x^4}}$

$u = x^2$

$du = 2x dx$

$\frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}$

$\frac{1}{2} \sin^{-1}(x^2) + C$

(f) $\int \frac{x+3}{x^2-1} dx = \int \frac{x+3}{(x-1)(x+1)} dx$

$2 \int \frac{dx}{x-1} - 1 \int \frac{dx}{x+1}$

$2 \ln|x-1| - \ln|x+1| + C$

Bonus: $\int \frac{\cos^3 x dx}{\sqrt{\sin x}} = \int \frac{\cos x (1 - \sin^2 x) dx}{(\sin x)^{\frac{1}{2}}} \frac{1}{(\sin x)^{\frac{1}{2}}}$

$u = \sin x$
 $1 - \sin^2 x = \cos^2 x$

$= 2 \sqrt{\sin x} - \frac{2}{5} \sin^{\frac{5}{2}} x + C$

or
 $\ln \left| \frac{(x-1)^2}{x+1} \right| + C$

1. Determine whether the series converges or diverges. (state convergence test used)

$$(a) \sum_{n=1}^{\infty} \frac{n-1}{2n-1}$$

Diverge, n^{th} term test for divergence

$$(d) \sum_{k=1}^{\infty} \frac{2^k}{k!}$$

converge, Ratio test

$$(b) \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Converge, Limit Comp. test
to $\lesssim \frac{1}{n^2}$

$$(e) \sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Converge,
Direct comp
to $\lesssim (\frac{1}{3})^n$

$$(c) \sum_{n=1}^{\infty} n e^{-n^2}$$

converge, Integral Test

$$(f) \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$$

converge, Alt. Ser. Test

$$(g) \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

Diverge, direct comp
 $\sum \frac{1}{n}$

$$(h) \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$$

Converge, Direct comp
 $\sum \frac{5}{n^2}$

$$(i) \sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2} \right)^n$$

converge, Root test

$$(j) \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

* Diverge, Ratio Test

$$(k) \sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

Converge, Direct comp
 $\sum \frac{1}{n^2}$

2. Find the Maclaurin series for the function $f(x) = \frac{1}{\sqrt{4-x}}$ and its radius of convergence.

$$f(x) = (4-x)^{-\frac{1}{2}} \quad f(0) = \frac{1}{2}$$

$$f(x) = \frac{1}{2} + \frac{1}{2^4} \left(\frac{x}{2}\right) + \frac{3}{2^7} \left(\frac{x^2}{2^2}\right) + \frac{5 \cdot 3 \cdot 1}{2^{10}} \left(\frac{x^3}{2^3}\right) + \dots$$

$$f'(x) = \frac{1}{2} (4-x)^{-\frac{3}{2}} \quad f'(0) = \frac{1}{2^4}$$

$$f''(x) = \frac{3}{2^2} (4-x)^{-\frac{5}{2}} \quad f''(0) = \frac{3}{2^2} \left(\frac{1}{2^5}\right) = \frac{3}{2^7}$$

$$+ \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^{13}} \cdot \left(\frac{x^4}{4!}\right) + \dots$$

$$f'''(x) = \frac{5 \cdot 3 \cdot 1}{2^3} (4-x)^{-\frac{7}{2}} \quad f'''(0) = \frac{5 \cdot 3 \cdot 1}{2^3} \left(\frac{1}{2^7}\right) = \frac{5 \cdot 3 \cdot 1}{2^{10}}$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(2n-1) \dots 5 \cdot 3 \cdot 1}{2^{3n+1} (n!)} x^n$$

$$f^{IV}(x) = \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4} (4-x)^{-\frac{9}{2}} \quad f^{IV}(0) = \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^4} \left(\frac{1}{2^9}\right) = \frac{7 \cdot 5 \cdot 3 \cdot 1}{2^{13}}$$

$$r = 4$$

3. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x-3)^{n+1}}{n+1}}{\frac{(x-3)^n}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)n}{n+1} \right| = |x-3| < 1 \quad -1 < x-3 < 1$$

$$2 < x < 4$$

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = \text{Diverge, Harmonic series } \therefore [2, 4)$$

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} \text{ is a Harmonic alternating function.}$$