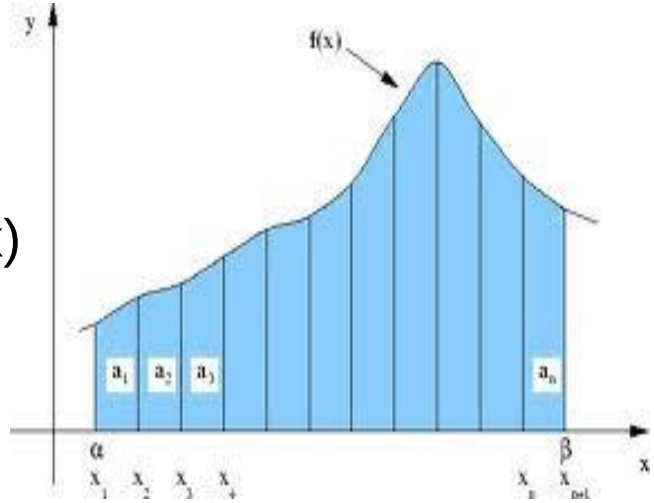
Finding Area inside Polar Regions

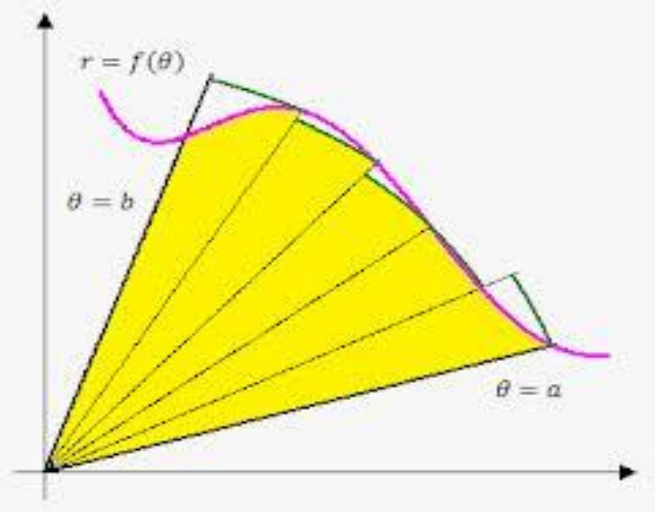
Remember to find area under a curve you add up an infinite number of rectangles

Area of a rectangle = length (f(x)) * width (dx)



$$\int_a^b f(x)\,dx$$

To find area enclosed by a polar curve, you add up an infinite number of sectors.



Area of a sector = fraction of a circle

$$= \frac{\theta}{2\pi} * \pi r^2$$

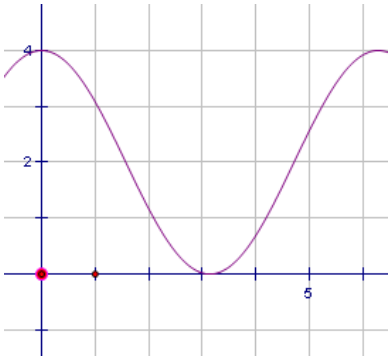
$$= \frac{1}{2} r^2 \theta$$

Area enclosed by a polar region $= \int_a^b \frac{1}{2} r^2 d\theta$

Example 1:

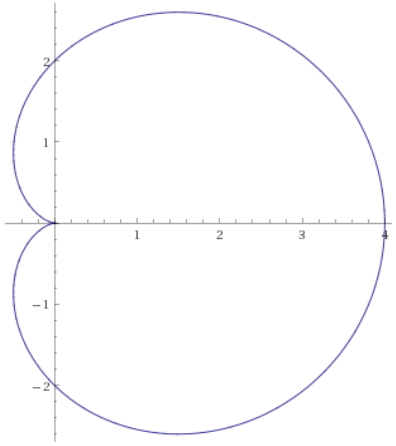
Find the area enclosed by $r = 2 + 2\cos\theta$

First sketch the graph to find what interval makes a full curve either by hand or on the calculator.



This curve is drawn from 0 to 2π.

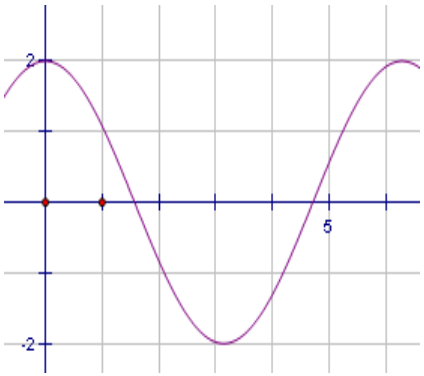
$$\frac{1}{2}\int_0^{2\pi}\left(2+2\cos\theta\right)^2 d\theta$$ =18.849 = 6π

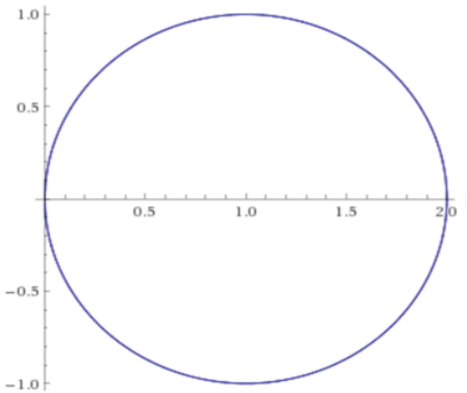


Example 2:
Find the area within $r = 2\cos\theta$

$$Area = \frac{1}{2}\int_0^{\pi} (2\cos\theta)^2 \, d\theta$$

$$Area = 3.1415 = \pi$$

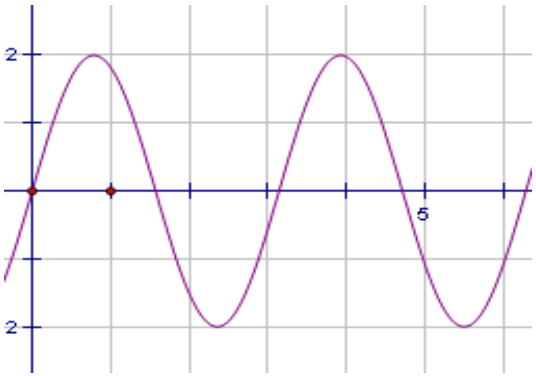




Make a full circle in π

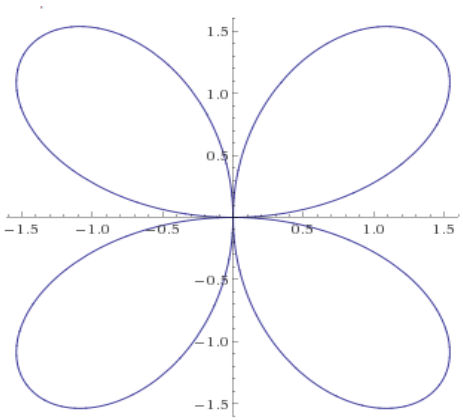
Example 3:
Find the area inside $r = 2\sin(2\theta)$

Use symmetry; find the area of one petal, them multiply by 4.



One petal is drawn from 0 to π/2

$$Area = 4 \bullet \frac{1}{2}\int_0^{\pi/2} \left[2\sin(2\theta)\right]^2 d\theta$$



Area = 6.283 or 2π

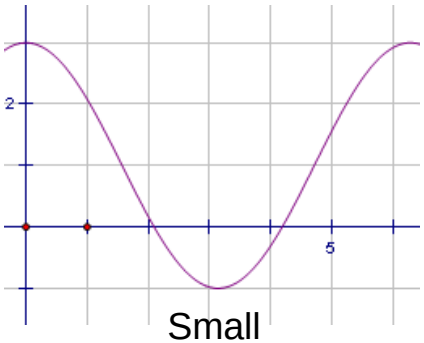
Example 4: Find the area inside the small loop of $r = 1 + 2\cos\theta$

The radius is zero at the beginning and end of the small loop.

$$1 + 2\cos\theta = 0$$

$$\cos\theta = \frac{-1}{2}$$

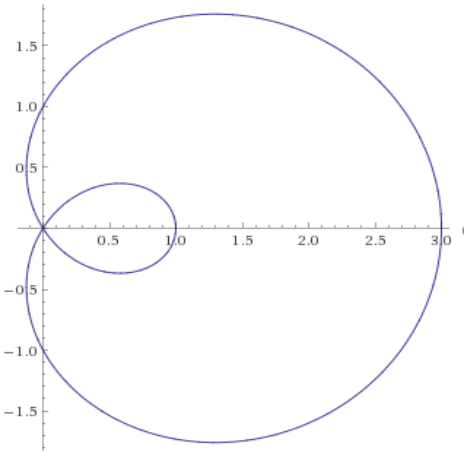
$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$



Small loop

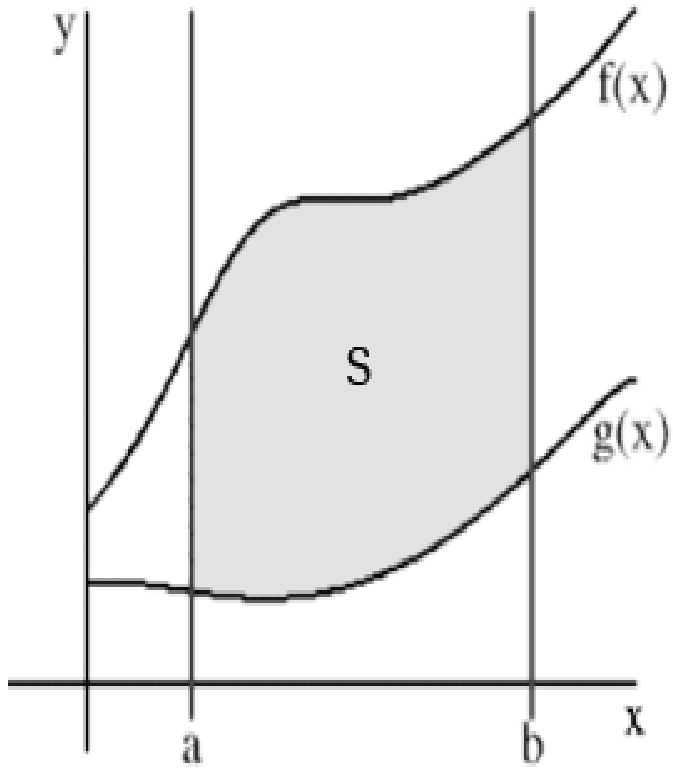
$$\frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 2\cos\theta)^2 \, d\theta$$

Area = 0.544



Homework: Sec 9.5 p.676 #1-17 odd

Remember to find the area between curves on a rectangular grid:

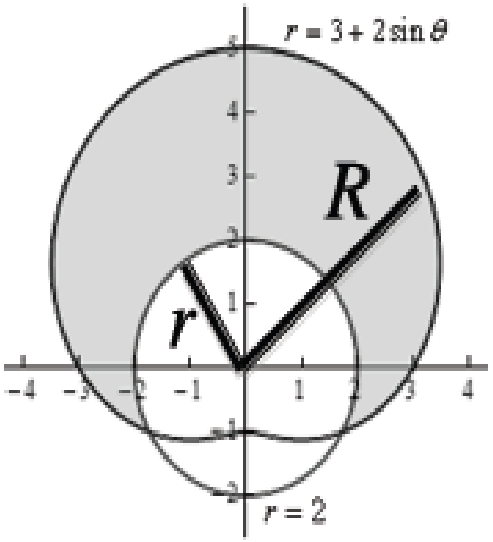


$$S = \int_a^b top - bottom$$

$$S = \int_a^b f(x) - g(x)\, dx$$

To find the area between polar curves:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} R^2 - r^2 \, d\theta$$



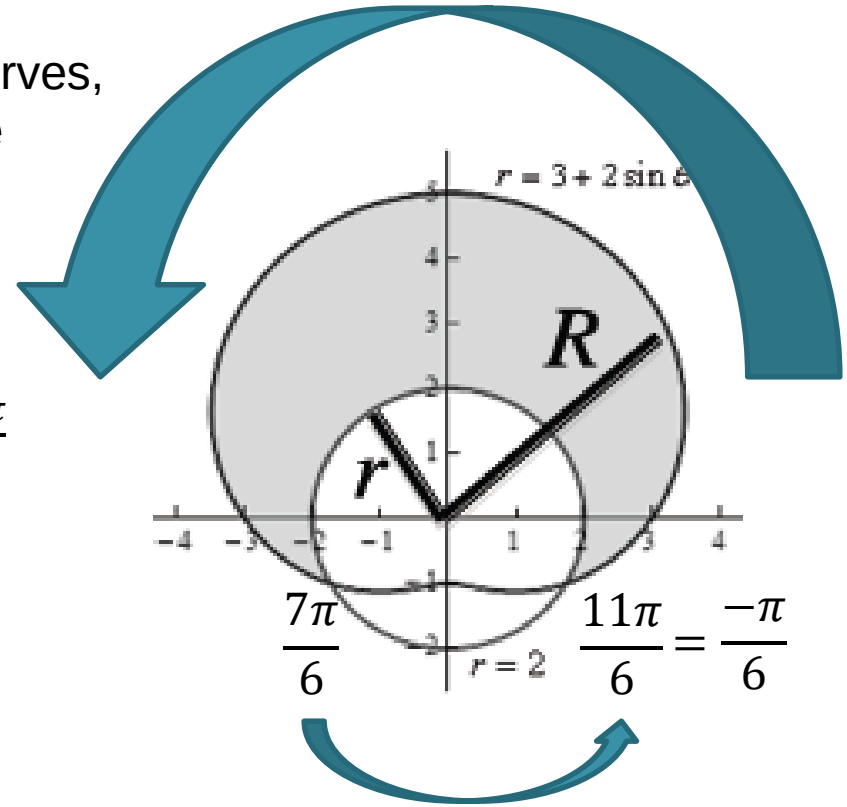
Example 5: Find the area inside $r = 3 + 2\sin\theta$ but outside r = 2

As with area between Cartesian curves, we need find the intersection of the curves.

$$3 + 2\sin\theta = 2$$
$$\sin\theta = -1/2$$
$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

We want a cardioid that starts at $\frac{-\pi}{6}$ and ends at $\frac{7\pi}{6}$.

Check that the circle has the same start and stop angles.

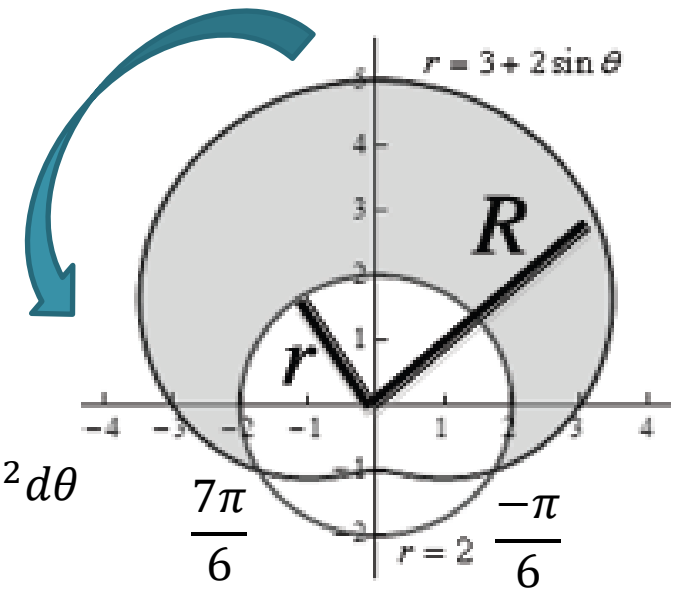


This is not the region we want.

Example 5: Find the area inside $r = 3 + 2\sin\theta$ but outside r = 2

We want a cardioid that starts at $\frac{-\pi}{6}$ and ends at $\frac{7\pi}{6}$.

Check that the circle has the same start and stop angles.

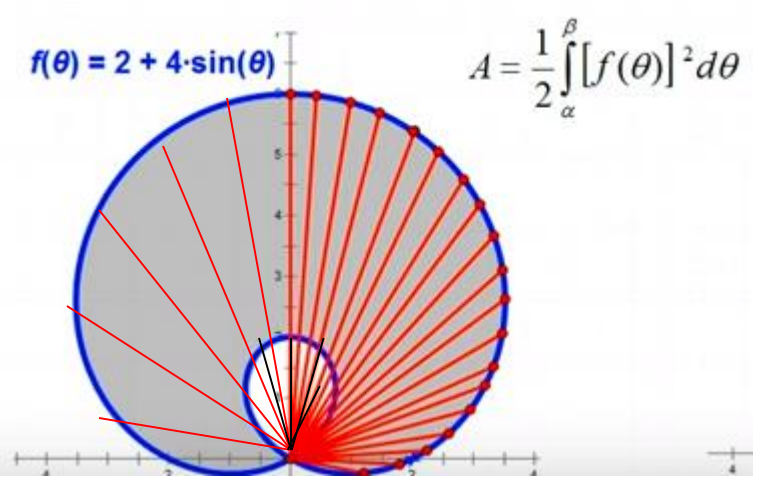


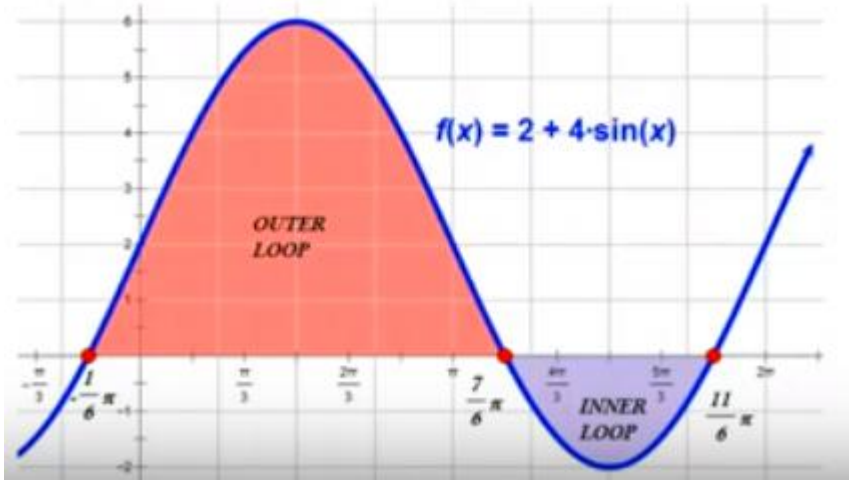
$$1/2 \int_{-\pi/6}^{7\pi/6} (3 + 2\sin\theta)^2 d\theta - 1/2 \int_{-\pi/6}^{7\pi/6} (2)^2 d\theta$$

$$1/2 \int_{-\pi/6}^{7\pi/6} (3 + 2\sin\theta)^2 - 2^2 d\theta$$

Or use symmetry. $2 \cdot \frac{1}{2} \int_{\pi/2}^{7\pi/6} (3 + 2\sin\theta)^2 - 2^2 d\theta$

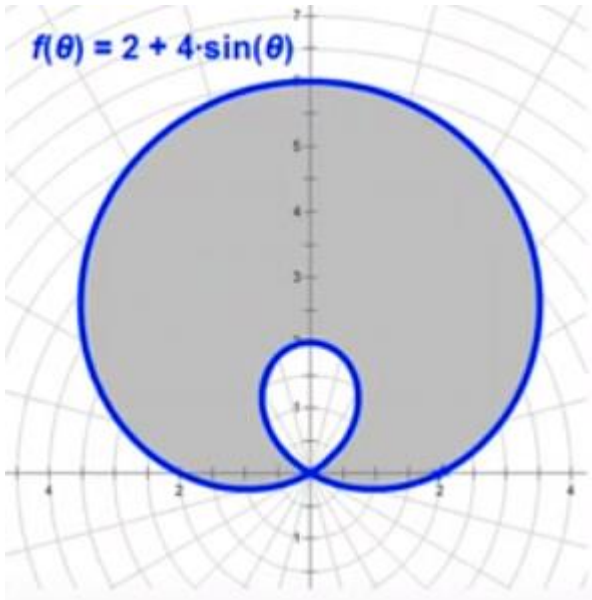
Example 6:Find the area between the loops of $r = 2 + 4\sin\theta$

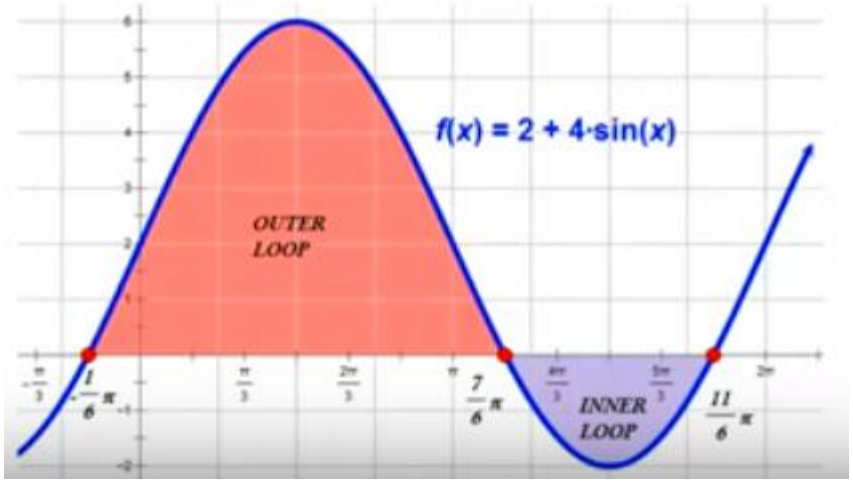




Be careful! $\int_0^{2\pi} 2 - 4\cos\theta \ d\theta$= inner + outer loop .

The inner loop lays on top of outer loop.





$$2 + 4\sin\theta = 0$$
$$\sin\theta = -1/2$$
$$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Large loop starts at $\frac{-\pi}{6}$ and stops at $\frac{7\pi}{6}$, or half the large loop starts at $\frac{\pi}{2}$ and stops at $\frac{7\pi}{6}$.

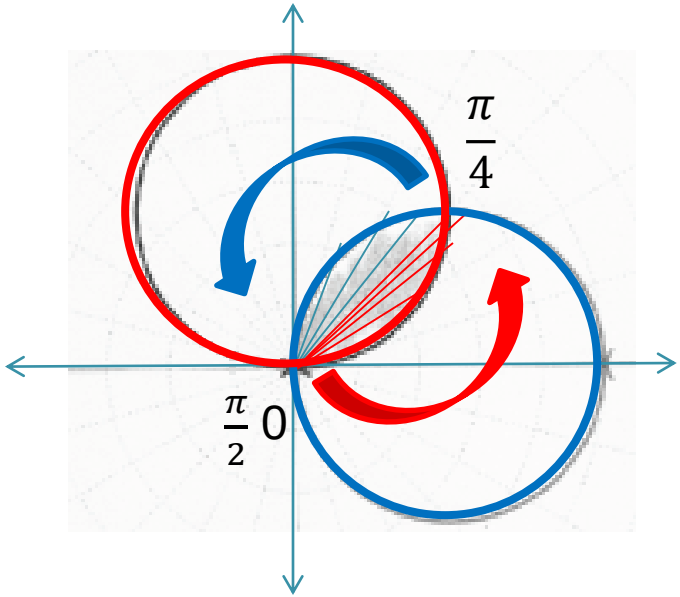
Small loop starts at $\frac{7\pi}{6}$ and stops at $\frac{11\pi}{6}$.

$$\frac{1}{2}\int_{-\pi/6}^{7\pi/6} (2 + 4sin\theta)^2 d\theta - \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (2 + 4sin\theta)^2 d\theta$$

=33.351

Example 7:

Find the area enclosed by r = sin θ and r = cos θ .



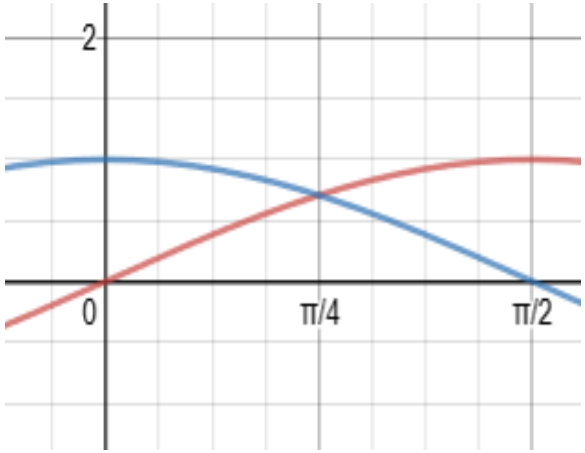
The curves intersect at $\theta = \frac{\pi}{4}$, when $\sin\theta = \cos\theta$

The area is shaded from the center to red curve until $\theta = \frac{\pi}{4}$, then switches to the blue curve.

$$\frac{1}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^2\theta\, d\theta$$

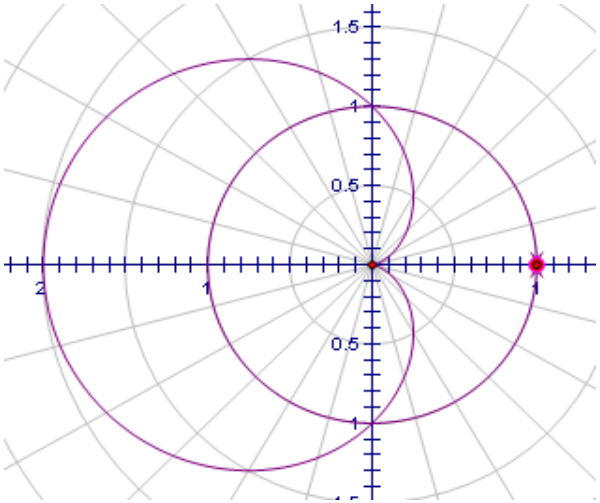
Or use symmetry and double the red section $2 \cdot \frac{1}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta$

=0.142699



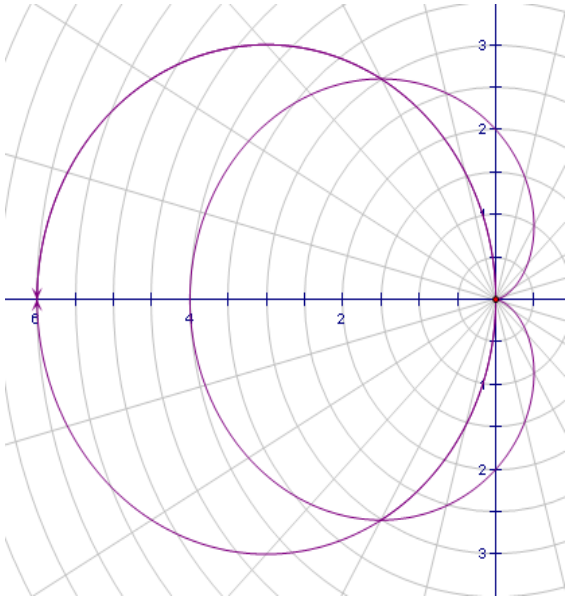
Try:

1. Find the area of the region inside r = 1 and outside
   r = 1 – cos Θ .



Answer: 1.125

2. Find the area of the region inside r = – 6 cos Θ and
   r = 2 – 2 cos Θ .



Answer: 15.708

Practice.

- Sec 9.5 p.676 #12, 14, 19 – 25odd, 37 - 45 odd, 77, 78