23. Integrate $\int \theta e^{\theta}\,d\theta$ by parts.

$u = \theta \qquad dv = e^{\theta}\,d\theta$

$du = d\theta \qquad v = e^{\theta}$

$\int \theta e^{\theta}\,d\theta = \theta e^{\theta} - \int e^{\theta}\,d\theta = \theta e^{\theta} - e^{\theta} + C$

$$\int_{-\infty}^{0} \theta e^{\theta}\,d\theta = \lim_{b\to-\infty} \int_{b}^{0} \theta e^{\theta}\,d\theta$$
$$= \lim_{b\to-\infty} \left[\theta e^{\theta} - e^{\theta}\right]_{b}^{0}$$
$$= \lim_{b\to-\infty} (-1 - be^{b} + e^{b}) = -1$$

$\left(\text{Note that } \lim_{b\to-\infty} be^{b} = \lim_{c\to\infty} -ce^{-c} = \lim_{c\to\infty} -\frac{c}{e^{c}}\right.$
$\left.= \lim_{c\to\infty} -\frac{1}{e^{c}} = 0 \text{ and } \lim_{b\to-\infty} e^{b} = \lim_{c\to\infty} e^{-c} = 0.\right)$

24. Integrate $\int 2e^{-\theta}\sin\theta\,d\theta$ by parts.

$u = 2\sin\theta \qquad dv = e^{-\theta}\,d\theta$

$du = 2\cos\theta\,d\theta \qquad v = -e^{-\theta}$

$\int 2e^{-\theta}\sin\theta\,d\theta = -2e^{-\theta}\sin\theta + \int 2e^{-\theta}\cos\theta\,d\theta$

Integrate $\int 2e^{-\theta}\cos\theta\,d\theta$ by parts.

$u = 2\cos\theta \qquad dv = e^{-\theta}\,d\theta$

$du = -2\sin\theta\,d\theta \qquad v = -e^{-\theta}$

$\int 2e^{-\theta}\cos\theta\,d\theta = -2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\,d\theta$

Thus,

$$\int 2e^{-\theta}\sin\theta\,d\theta$$
$$= -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta - \int 2e^{-\theta}\sin\theta\,d\theta$$
$$2\int 2e^{-\theta}\sin\theta\,d\theta = -2e^{-\theta}\sin\theta - 2e^{-\theta}\cos\theta + C_1$$
$$\int 2e^{-\theta}\sin\theta\,d\theta = -e^{-\theta}\sin\theta - e^{-\theta}\cos\theta + C$$
$$\int_{0}^{\infty} 2e^{-\theta}\sin\theta\,d\theta = \lim_{b\to\infty}\int_{0}^{b} 2e^{-\theta}\sin\theta\,d\theta$$
$$= \lim_{b\to\infty}\left[-e^{-\theta}\sin\theta - e^{-\theta}\cos\theta\right]_{0}^{b}$$
$$= \lim_{b\to\infty}(-e^{-b}\sin b - e^{-b}\cos b + 1) = 1$$

25. $\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx$

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{b\to-\infty}\int_{b}^{0} e^{x}\,dx = \lim_{b\to-\infty}\left[e^{x}\right]_{b}^{0} = \lim_{b\to-\infty}(1 - e^{b}) = 1$$
$$\int_{0}^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\int_{0}^{b} e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_{0}^{b}$$
$$= \lim_{b\to\infty}(-e^{-b} + 1) = 1$$
$$\int_{-\infty}^{\infty} e^{-|x|}\,dx = 1 + 1 = 2$$

26. Integrate $\int x\ln x\,dx$ by parts.

$u = \ln x \qquad dv = x\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{2}x^2$

$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

$$\int_{0}^{1} x\ln x = \lim_{b\to0^+}\int_{b}^{1} x\ln x\,dx$$
$$= \lim_{b\to0^+}\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_{b}^{1}$$
$$= \lim_{b\to0^+}\left(-\frac{1}{4} - \frac{1}{2}b^2\ln b + \frac{1}{4}b^2\right)$$
$$= -\frac{1}{4}$$

$\left(\text{Note that } \lim_{b\to0^+} b^2\ln b = \lim_{b\to0^+}\frac{\ln b}{1/b^2} = \lim_{b\to0^+}\frac{1/b}{-2/b^3}\right.$
$\left.= \lim_{b\to0^+} -\frac{b^2}{2} = 0.\right)$

27. $\int_{0}^{\pi/2} \tan\theta\,d\theta = \lim_{b\to\pi/2}\int_{0}^{b}\frac{\sin\theta}{\cos\theta}\,d\theta$
$$= \lim_{b\to\pi/2}\left[-\ln|\cos\theta|\right]_{0}^{b}$$
$$= \lim_{b\to\pi/2}[-\ln|\cos b| + 0] = \infty$$

The integral diverges.

28. On $[0, \pi]$, $0 \le \dfrac{\sin\theta}{\sqrt{\pi-\theta}} \le \dfrac{1}{\sqrt{\pi-\theta}}$, so

$$\int_0^{\pi} \frac{\sin\theta}{\sqrt{\pi-\theta}}\,d\theta \le \int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}}\,d\theta$$

$$\begin{aligned}\int_0^{\pi} \frac{1}{\sqrt{\pi-\theta}}\,d\theta &= \lim_{b\to\pi^-} \int_0^{b} \frac{1}{\sqrt{\pi-\theta}}\,d\theta \\ &= \lim_{b\to\pi^-} \left[-2\sqrt{\pi-\theta}\right]_0^b \\ &= \lim_{b\to\pi^-} (-2\sqrt{\pi-b} + 2\sqrt{\pi}) \\ &= -2\sqrt{0} + 2\sqrt{\pi} \\ &= 2\sqrt{\pi}\end{aligned}$$

Since this integral converges, the given integral converges.

29. $\displaystyle\int_{-\infty}^{\infty} 2xe^{-x^2}\,dx = \int_0^{\infty} 2xe^{-x^2}\,dx + \int_{-\infty}^{0} 2xe^{-x^2}\,dx$

$$\begin{aligned}\int_0^{\infty} 2xe^{-x^2}\,dx &= \lim_{b\to\infty} \int_0^{b} 2xe^{-x^2}\,dx \\ &= \lim_{b\to\infty} \left[-e^{-x^2}\right]_0^b \\ &= \lim_{b\to\infty} [-e^{-b^2} + 1] = 1\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{0} 2xe^{-x^2}\,dx &= \lim_{b\to-\infty} \int_b^{0} 2xe^{-x^2}\,dx \\ &= \lim_{b\to-\infty} \left[-e^{-x^2}\right]_b^0 \\ &= \lim_{b\to-\infty} [-1 + e^{-b^2}] = -1\end{aligned}$$

The integral converges.

30.
$$\begin{aligned}\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \lim_{b\to0^+} \int_b^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \\ &= \lim_{b\to0^+} \left[-2e^{-\sqrt{x}}\right]_b^4 \\ &= \lim_{b\to0^+} [-2e^{-2} + 2e^{-\sqrt{b}}] \\ &= -2e^{-2} + 2\end{aligned}$$

The integral converges.

Converges

31. $0 \le \dfrac{1}{\sqrt{t} + \sin t} \le \dfrac{1}{\sqrt{t}}$ on $(0, \pi]$ since $\sin t \ge 0$ on $[0, \pi]$.

$$\begin{aligned}\int_0^{\pi} \frac{dt}{\sqrt{t}} &= \lim_{b\to0^+} \int_b^{\pi} \frac{dt}{\sqrt{t}} \\ &= \lim_{b\to0^+} \left[2\sqrt{t}\right]_b^{\pi} \\ &= \lim_{b\to0^+} [2\sqrt{\pi} - 2\sqrt{b}] \\ &= 2\sqrt{\pi}\end{aligned}$$

smaller

Since this integral converges, the given integral converges.

32. $0 \le \dfrac{1}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}-1}$ on $[4, \infty)$

$$\int_4^{\infty} \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \int_4^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty} \left[2\sqrt{x}\right]_4^b = \lim_{b\to\infty} [2\sqrt{b} - 4] = \infty$$

Since this integral diverges, the given integral diverges.

Con.

33. $0 \le \dfrac{1}{x^3+1} \le \dfrac{1}{x^3}$ on $[1, \infty)$

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^3} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} \\ &= \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^b \\ &= \lim_{b\to\infty} \left[-\frac{1}{2}b^{-2} + \frac{1}{2}\right] = \frac{1}{2}\end{aligned}$$

smaller

Since this integral converges, the given integral converges.

34. $\displaystyle\int_0^2 \frac{dx}{1-x^2} = \int_0^1 \frac{dx}{1-x^2} + \int_1^2 \frac{dx}{1-x^2}$

$$\begin{aligned}\int_0^1 \frac{dx}{1-x^2} &= \lim_{b\to1^-} \int_0^b \frac{1/2[(1-x)+(1+x)]}{(1-x)(1+x)}\,dx \\ &= \lim_{b\to1^-} \int_0^b \left[\frac{1}{2(1+x)} + \frac{1}{2(1-x)}\right] dx \\ &= \lim_{b\to1^-} \left[\frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\right]_0^b \\ &= \lim_{b\to1^-} \left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^b \\ &= \lim_{b\to1^-} \left[\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right| - 0\right] = \infty\end{aligned}$$

Since this integral diverges, the given integral diverges.

35. $\displaystyle\int_0^2 \frac{dx}{1-x} = \int_0^1 \frac{dx}{1-x} + \int_1^2 \frac{dx}{1-x}$

$$\begin{aligned}\int_0^1 \frac{dx}{1-x} &= \lim_{b\to1^-} \int_0^b \frac{dx}{1-x} \\ &= \lim_{b\to1^-} \left[-\ln|1-x|\right]_0^b \\ &= \lim_{b\to1^-} (-\ln|1-b| + 0) = \infty\end{aligned}$$

Since this integral diverges, the given integral diverges.

Part whole

36. $\int_{-1}^{1} \ln|x|\,dx = 2\int_0^1 \ln x\,dx$ by symmetry of $\ln|x|$ about the y-axis. Integrate $\int \ln x\,dx$ by parts.

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\,dx \qquad v = x$$

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C$$
$$2\int_0^1 \ln x\,dx = 2\lim_{b\to 0^+}\int_b^1 \ln x\,dx$$
$$= 2\lim_{b\to 0^+}\Big[x\ln x - x\Big]_b^1$$
$$= 2\lim_{b\to 0^+}[-1 - b\ln b + b] = -2$$

$\left(\text{Note that } \lim_{b\to 0^+} b\ln b = \lim_{b\to 0^+}\frac{\ln b}{1/b} = \lim_{b\to 0^+}\frac{1/b}{-1/b^2}\right.$
$\left. = \lim_{b\to 0^+} -b = 0.\right)$

The integral converges.

37. $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ on $[1, \infty)$

$$\int_1^\infty \frac{1}{e^\theta}\,d\theta = \lim_{b\to\infty}\int_1^b e^{-\theta}\,d\theta$$
$$= \lim_{b\to\infty}\Big[-e^{-\theta}\Big]_1^b$$
$$= \lim_{b\to\infty}[-e^{-b} + e^{-1}]$$
$$= \frac{1}{e}$$

smaller

Since this integral converges, the given integral converges.

38. $0 \le \frac{1}{x} \le \frac{1}{\sqrt{x^2-1}}$ on $[2, \infty)$

$$\int_2^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_2^b \frac{dx}{x} = \lim_{b\to\infty}\Big[\ln x\Big]_2^b = \lim_{b\to\infty}(\ln b - \ln 2) = \infty$$

Since this integral diverges, the given integral diverges.

39. Let $f(x) = \frac{\sqrt{x+1}}{x^2}$ and $g(x) = \frac{1}{x^{3/2}}$. Both are continuous on $[1, \infty)$.

$x^{\frac{1}{2}-2}$

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{x}} = \lim_{x\to\infty}\sqrt{1+\frac{1}{x}} = 1$$
$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b x^{-3/2}\,dx$$
$$= \lim_{b\to\infty}\Big[-2x^{-1/2}\Big]_1^b$$
$$= \lim_{b\to\infty}(-2b^{-1/2} + 2) = 2$$

Since this integral converges, the given integral converges.

40. $\int_0^\infty \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{\sqrt{x}} + \int_1^\infty \frac{dx}{\sqrt{x}}$.

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}}$$
$$= \lim_{b\to\infty}\Big[2\sqrt{x}\Big]_1^b$$
$$= \lim_{b\to\infty}(2\sqrt{b} - 2) = \infty$$

Since this integral diverges, the given integral diverges.

41. $0 \le \frac{1}{x} \le \frac{2+\cos x}{x}$ on $[\pi, \infty)$.

$$\int_\pi^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_\pi^b \frac{dx}{x} = \lim_{b\to\infty}\Big[\ln x\Big]_\pi^b = \lim_{b\to\infty}(\ln b - \ln\pi) = \infty$$

Since this integral diverges, the given integral diverges.

42. $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ on $[\pi, \infty)$

$$\int_\pi^\infty \frac{2\,dx}{x^2} = \lim_{b\to\infty}\int_\pi^b 2x^{-2}\,dx$$
$$= \lim_{b\to\infty}\Big[-2x^{-1}\Big]_\pi^b$$
$$= \lim_{b\to\infty}\left(-2b^{-1} + \frac{2}{\pi}\right) = \frac{2}{\pi}$$

Since this integral converges, the given integral converges.

43. First rewrite $\frac{1}{e^x + e^{-x}}$.

$$\frac{1}{e^x + e^{-x}} = \frac{1}{e^{-x}(e^{2x}+1)} = \frac{e^x}{1+(e^x)^2}$$

Integrate $\int \frac{e^x\,dx}{1+(e^x)^2}$ by letting $u = e^x$ so $du = e^x\,dx$.

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{1+(e^x)^2}$$
$$= \int \frac{du}{1+u^2}$$
$$= \tan^{-1} u + C$$
$$= \tan^{-1} e^x + C$$

$$\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}} = \int_{-\infty}^0 \frac{dx}{e^x+e^{-x}} + \int_0^\infty \frac{dx}{e^x+e^{-x}}$$
$$\int_{-\infty}^0 \frac{dx}{e^x+e^{-x}} = \lim_{b\to-\infty}\int_b^0 \frac{dx}{e^x+e^{-x}}$$
$$= \lim_{b\to-\infty}\Big[\tan^{-1} e^x\Big]_b^0$$
$$= \lim_{b\to-\infty}[\tan^{-1} 1 - \tan^{-1} e^b]$$
$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$
$$\int_0^\infty \frac{dx}{e^x+e^{-x}} = \lim_{b\to\infty}\int_0^b \frac{dx}{e^x+e^{-x}}$$
$$= \lim_{b\to\infty}\Big[\tan^{-1} e^x\Big]_0^b$$
$$= \lim_{b\to\infty}[\tan^{-1} e^b - \tan^{-1} 1]$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus, the given integral converges.

44. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+1}} = 2\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}}$ by symmetry about the y-axis

$$\int_0^{\infty} \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+1}}$$

$\int_0^1 \frac{dx}{\sqrt{x^4+1}}$ exists because $\frac{1}{\sqrt{x^4+1}}$ exists on $[0, 1]$.

$0 \le \frac{1}{\sqrt{x^4+1}} \le \frac{1}{x^2}$ on $[1, \infty)$.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx$$
$$= \lim_{b\to\infty} \left[-x^{-1}\right]_1^b$$
$$= \lim_{b\to\infty} \left[-\frac{1}{b} + 1\right] = 1$$

Since this integral converges, the given integral converges.

45. Integrate $\int \frac{dy}{(1+y^2)(1+\tan^{-1} y)}$ by letting $u = \tan^{-1} y$ so

$du = \frac{dy}{1+y^2}$

$$\int \frac{dy}{(1+y^2)(1+\tan^{-1} y)} = \int \frac{du}{1+u}$$

1 + u

$$= \ln|1+u| + C$$
$$= \ln|1+\tan^{-1} y| + C$$

$$\int_0^{\infty} \frac{dy}{(1+y^2)(1+\tan^{-1} y)} = \lim_{b\to\infty} \int_0^b \frac{dy}{(1+y^2)(1+\tan^{-1} y)}$$
$$= \lim_{b\to\infty} \left[\ln|1+\tan^{-1} y|\right]_0^b$$
$$= \lim_{b\to\infty} (\ln|1+\tan^{-1} b| - 0)$$
$$= \ln\left(1+\frac{\pi}{2}\right)$$

The integral converges.

46. $\int_{-\infty}^{\infty} \frac{e^{-y}\,dy}{y^2+1} = \int_{-\infty}^{0} \frac{e^{-y}\,dy}{y^2+1} + \int_0^{\infty} \frac{e^{-y}\,dy}{y^2+1}$

$\int_{-\infty}^{0} \frac{e^{-y}\,dy}{y^2+1}$ diverges since

$$\lim_{y\to-\infty} \frac{e^{-y}}{y^2+1} = \lim_{y\to\infty} \frac{e^{y}}{y^2+1} = \lim_{y\to\infty} \frac{e^y}{2y} = \lim_{y\to\infty} \frac{e^y}{2} = \infty$$

Thus the given integral diverges.

47. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

$$\text{Area} = \int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

Integrate $\int \frac{\ln x}{x^2}\,dx$ by parts.

$u = \ln x \qquad dv = \frac{dx}{x^2}$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\text{Area} = \lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] = 1$$

$\left(\text{Note that } \lim_{b\to\infty} \frac{\ln b}{b} = \lim_{b\to\infty} \frac{1/b}{1} = 0.\right)$

48. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

$$\text{Area} = \int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx$$

Integrate $\int \frac{\ln x}{x}\,dx$ by letting $u = \ln x$ so $du = \frac{dx}{x}$.

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

$$\text{Area} = \lim_{b\to\infty} \left[\frac{1}{2}(\ln x)^2\right]_1^b = \lim_{b\to\infty} \frac{1}{2}(\ln b)^2 = \infty$$

49. (a) The integral in Example 1 gives the area of region R.

$$\text{Area} = \int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big]_1^b = \infty$$

(b) Refer to Exploration 2 of Section 7.3.

$y' = -\frac{1}{x^2}$

The surface area of the solid is given by the following integral.

$$\int_1^{\infty} 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx = 2\pi \int_1^{\infty} \frac{1}{x}\sqrt{\frac{x^4+1}{x^4}}\,dx$$
$$= 2\pi \int_1^{\infty} \frac{\sqrt{x^4+1}}{x^3}\,dx$$

Since $0 \le \frac{1}{x} \le \frac{\sqrt{x^4+1}}{x^3}$ on $[1, \infty)$, the direct comparison test shows that the integral for the surface area diverges. The surface area is ∞.

(c) $$\text{Volume} = \int_1^{\infty} \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^{\infty} \frac{1}{x^2}\,dx$$
$$= \pi \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$$
$$= \pi \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b$$
$$= \pi \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = \pi$$

(d) Gabriel's horn has finite volume so it could only hold a finite amount of paint, but it has infinite surface area so it would require an infinite amount of paint to cover itself.