Length of Polar Curves

Review length of parametric equations

$$\int \sqrt{\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2}\, d\theta$$

So given a polar function $r = r(\theta)$, $x = r\cos\theta$ and $y = r\sin\theta$

$$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$$

$$\frac{dx}{d\theta}^2 + \frac{dy}{d\theta}^2 = (r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$$

$$\frac{dx^2}{d\theta} + \frac{dy^2}{d\theta} = (r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2$$

$$= (r'\cos\theta)^2 - 2r'r\cos\theta\sin\theta + (r\sin\theta)^2 + (r'\sin\theta)^2 + 2r'r\cos\theta\sin\theta + (r\cos\theta)^2$$

$$= (r')^2(\cos\theta)^2 + (r)^2(\sin\theta)^2 + (r')^2(\sin\theta)^2 + (r)^2(\cos\theta)^2$$

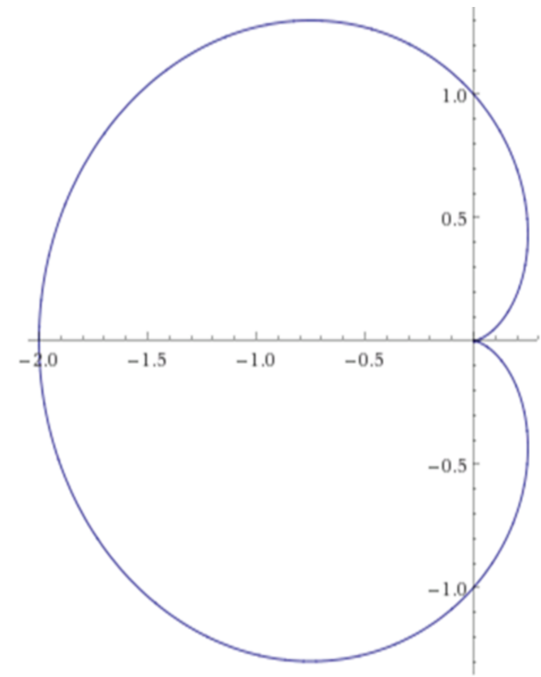
$$= (r')^2\underbrace{[\cos^2\theta + \sin^2\theta]}_{=1} + r^2\underbrace{[\cos^2\theta + \sin^2\theta]}_{=1}$$

$$= (r')^2 + r^2$$

Therefore length of a polar equation = $\int \sqrt{(r')^2 + r^2} d\theta$

Example 1: Find the length of the cardiod $r = 1 - \cos\theta$. $\int \sqrt{(r')^2 + r^2}\,d\theta$

The graph starts at $\theta = 0$ and ends at $\theta = 2\pi$.



$$r' = \sin\theta$$

So the length $= \int_0^{2\pi} \sqrt{(sin\theta)^2 + (1 - \cos\theta)^2}\,d\theta = 8$ units

Calculator short cut:

In Polar mode, Enter function in **Y=**   r1 =1 – cos ϴ

Then enter the derivative in **Y =** , but have the calculator do the derivative for you.

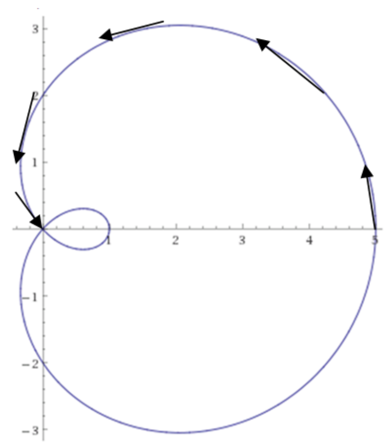
r2 = nDeriv(r1, ϴ, ϴ)   nDerive is **Math** 8,

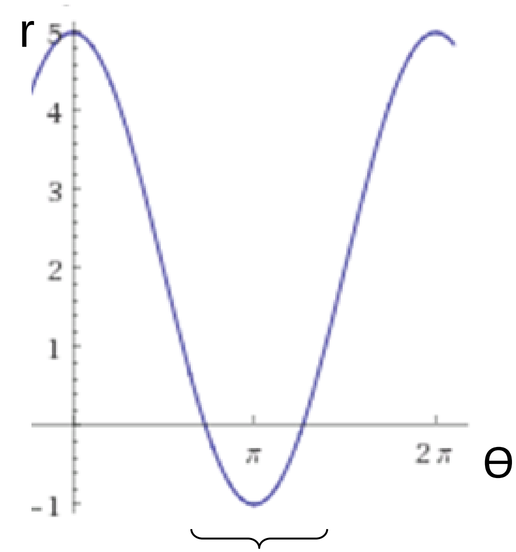
Then integrate the length formula using **Math** 9 using r1 and r2 for r and r’

$$\text{fnINT}(\sqrt{(\text{r1})^2+(r2)^2},\theta,0,2\pi)$$

r1 and r2 are found in **VARS**, Y-Vars, Polar

Example 2: Find the length of the outer loop of $r = 2 + 3\cos\theta$





Looking at the graph on the calculator, I can see starting at θ = 0 and stopping at the pole (when r = 0) takes me halfway around the outer loop.

Looking at the same graph on a rectangular r, θ grid, you can see the ½ the large loop above the axis, the small loop below the axis, then the rest of the large loop is above the axis.

Example 2: Find the length of the outer loop of $r = 2 + 3\cos\theta$

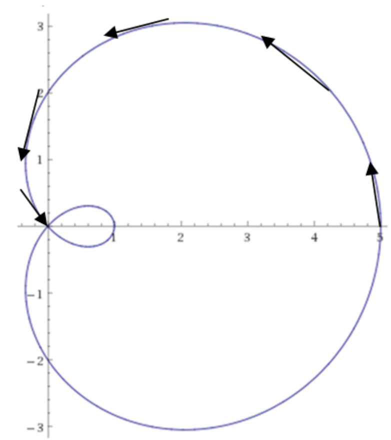
So, I need to calculate the exact angle value at the pole, since it is hard to trace to exactly r = 0.

$$r = 0 = 2 + 3\cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-2}{3}\right)$$

Store this value

$$r' = -3\sin\theta$$



so, the length of ½ the outer loop = $\int_0^{\cos^{-1}\left(\frac{-2}{3}\right)} \sqrt{(-3\sin\theta)^2 + (2+3\cos\theta)^2}\,d\theta = 9.252510543$

Whole outer loop = 18.505 units

Homework:
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