Unit 4
More with Derivatives

Section 7.7 Indeterminate forms and L’Hospital’s (or L’Hôpital’s) Rule

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x - 1}$$

Although F is not defined when $x = 1$, we need to know how F behaves near 1.

In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

$$f(x) = \frac{\ln x}{x - 1}$$

f(1) does not exist,
but let's look near 1

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \frac{0}{0}$$

Limit equals an indeterminate form

Maybe it approaches 0
Maybe it approaches ∞
Maybe it approaches some other number

We don't know . . . yet

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of *F* and need to evaluate the limit

$$\lim_{x\to\infty}\frac{\ln x}{x-1}=\frac{\infty}{\infty}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$.

There is a struggle between the two.

If the numerator wins, the limit will be ∞.

If the denominator wins, the answer will be 0.

Alternatively, there may be some compromise—
the answer may be some finite positive number.

Hence, in this section, we introduce a systematic method, known as l'Hospital's Rule, for the evaluation of indeterminate forms.

L’Hôpital’s Rule

Actually, L’Hôpital’s Rule was developed by his teacher Johann Bernoulli. De l’Hôpital paid Bernoulli for private lessons, and then published the first Calculus book based on those lessons.

Guillaume De l'Hôpital
1661 - 1704

Johann Bernoulli
1667 - 1748

L'Hôpital's Rule:

If f(x) and g(x) are differentiable and

$$\lim_{x\to 1}\frac{f(x)}{g(x)} = \frac{0}{0} \ or \ \frac{\pm\infty}{\pm\infty}$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

The rule is also valid for one-sided limits

We can confirm L'Hôpital's rule by working backwards, and using the definition of derivative:

If f(a) = g(a) = 0

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} \qquad = \lim_{x\to a}\frac{f(x)-0}{g(x)-0} \quad = \lim_{x\to a}\frac{f(x)}{g(x)}$$

It is more difficult to prove
the general version of l'Hospital's Rule.

Example 1

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \frac{0}{0}$$

Thus, we can apply l'Hospital's Rule:

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1}\frac{1/x}{1} = \lim_{x\to 1}\frac{1}{x} = 1$$

Example 2

Calculate $\lim_{x\to\infty}\frac{e^x}{x^2}=\frac{\infty}{\infty}$

So, l'Hospital's Rule gives:

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)}=\lim_{x\to\infty}\frac{e^x}{2x}=\frac{\infty}{\infty}$$

Example 2

$$\lim_{x\to\infty}\frac{e^x}{x^2}=\lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)}=\lim_{x\to\infty}\frac{e^x}{2x}=\frac{\infty}{\infty}$$

This limit is also indeterminate.

However, a second application of l'Hospital's Rule gives:

$$\lim_{x\to\infty}\frac{e^x}{x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{2x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{2}=\frac{\infty}{2}=\infty$$

Example 3

Find $\lim\limits_{x\to\pi^-} \dfrac{\sin x}{1-\cos x}$

If we blindly attempted to use l-Hospital's rule, we would get:

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \lim_{x\to\pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is wrong!

$$\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x} = \frac{\sin \pi}{1-\cos \pi} = \frac{0}{1-(-1)} = 0$$

Example 4

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \leftarrow \frac{0}{0}$$

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{2}}-1-\frac{1}{2}x}{x^2}$$

(Rewritten in exponential form.)

$$\overset{H}{=}\lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \leftarrow \frac{0}{0}$$

$$\overset{H}{=}\lim_{x\to 0}\frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} \leftarrow \text{not } \frac{0}{0}$$

$$=\frac{-\frac{1}{4}}{2}$$

$$=-\frac{1}{8}$$

→

L'Hôpital's rule can be used to evaluate other indeterminate forms besides $\frac{0}{0}$.

The following are also considered indeterminate:

$$\frac{\infty}{\infty} \qquad \infty \cdot 0 \qquad \infty - \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

The first one, $\frac{\infty}{\infty}$, can be evaluated just like $\frac{0}{0}$.

The others must be changed to fractions first.

→

INDETERMINATE FORM—TYPE $0 \cdot \infty$

- We can deal with it by writing the product *fg* as a quotient:

$$fg = \frac{f}{1/g} \quad \text{or} \quad fg = \frac{g}{1/f}$$

- This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or ∞/∞, so that we can use l'Hospital's Rule.

$$\lim_{x\to\infty}\left(x\sin\frac{1}{x}\right)$$ ← This approaches $\infty \cdot 0$

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}$$ ← This approaches $\frac{0}{0}$

if we want to use L'Hôpital's rule:

$$\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}} \overset{H}{=} \lim_{x\to\infty}\frac{\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\cos\left(\frac{1}{x}\right) = \cos(0) = 1$$

→

Indeterminate Form: $\infty - \infty$

Try to convert the difference into a quotient (for instance, by using a common denominator, rationalization, or factoring out a common factor) so that we have an indeterminate form of type 0/0 or ∞/∞.

Indeterminate Forms: 1^{∞} 0^{0} ∞^{0}

Evaluating these forms requires a mathematical trick to change the expression into a fraction. We will save that for next semester.