Warm-up

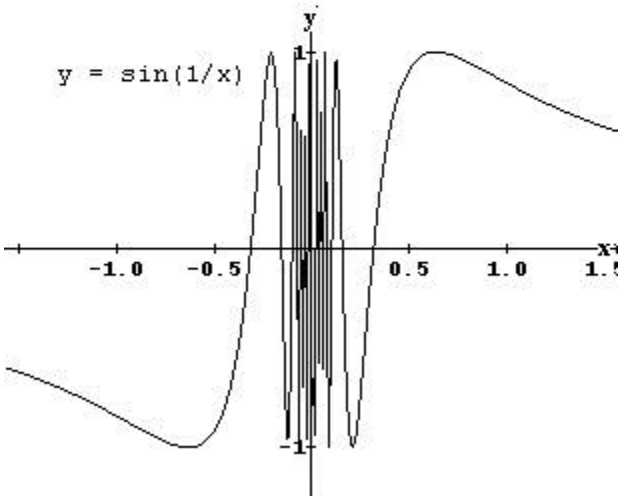
Use your calculator to evaluate the following limits.

1. $\lim_{x \to 0} \frac{\sin x}{x}$

2. $\lim_{x \to 0} \frac{1 - \cos x}{x}$

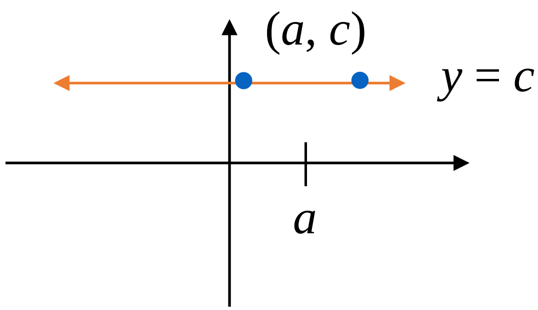
2.3 - Calculating Limits Using The Limit Laws

Tables and graphs are used to guess the values of limits, but such methods do not always lead to correct answers. So now we will use the laws of limits to calculate exact values.

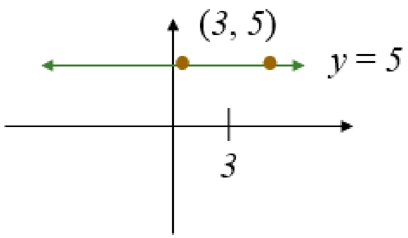


Basic Limit Laws

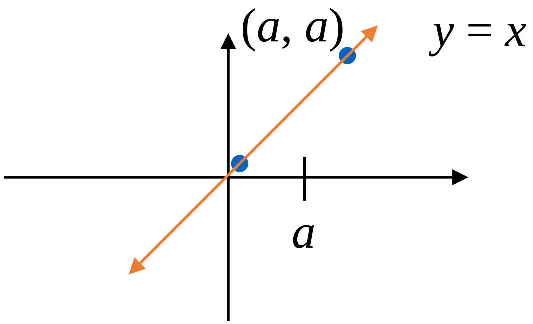
1. $\lim_{x \to a} c = c$



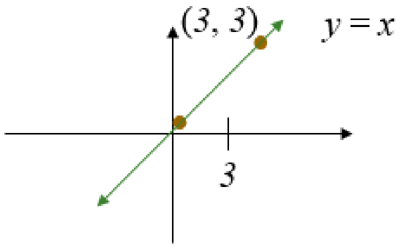
$\lim_{x \to 3} 5 = 5$



2. $\lim_{x \to a} x = a$



$\lim_{x \to 3} x = 3$



Limit Laws Generalized

Suppose that c is a constant and the following limits exist $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$.

$$1.\ \lim_{x \to a}\left[f(x)+g(x)\right]=\lim_{x \to a} f(x)+\lim_{x \to a} g(x)$$

$$2.\ \lim_{x \to a}\left[f(x)-g(x)\right]=\lim_{x \to a} f(x)-\lim_{x \to a} g(x)$$

$$3.\ \lim_{x \to a}\left[cf(x)\right]=c\lim_{x \to a} f(x)$$

Limit Laws Generalized

$$4.\ \lim_{x\to a}\left[f(x)g(x)\right]=\lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

$$5.\ \lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

$$6a.\ \lim_{x\to a}\left[f(x)\right]^n=\left[\lim_{x\to a} f(x)\right]^n$$ where n is a positive integer.

$$6b.\ \lim_{x\to a}\sqrt[n]{f(x)}=\sqrt[n]{\lim_{x\to a} f(x)}$$ where n is a positive integer.

Examples

Evaluate the following limits. Justify each step using the laws of limits.

1. $\lim_{x \to -3} \left(3x^2 + 2x - 5\right)$

2. $\lim_{x \to 1} \left(\frac{3x+2}{x-5}\right)$

3. $\lim_{x \to 2} \sqrt[3]{x^2 + 2x}$

Direct Substitution Property

If f is a *polynomial* or a *rational function* and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Examples

Evaluate the following limits using direct substitution.

1. $\lim_{x \to -3} \left(3x^2 + 2x - 5\right)$

2. $\lim_{x \to 1} \left(\dfrac{3x+2}{x-5}\right)$

3. $\lim_{x \to 2} \sqrt[3]{x^2 + 2x}$

You may encounter limit problems that seem to be impossible to compute or they appear to not exist. Here are some tricks to help you evaluate these limits.

1. If f is a ***rational function*** or complex:
   a. Simplify the function; eliminate common factors.
   b. Find a common denominator.
   c. Expand polynomials raised to a power.
2. If f is a ***root function***, rationalize the numerator.
3. If f is an ***absolute values function***, use one-sided limits and the definition:

$$|a| = \begin{cases} -a & \text{if } a < 0 \\ a & \text{if } a \geq 0 \end{cases}$$

4. If f is a ***trigonometric function***, use trig identities and the warm-up limits to write the function in a different form.

Direct Substitution Property

Evaluate the following limits, if they exist.

1. $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$

2. $\lim_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h}$

3. $\lim_{t \to 3} \dfrac{t - 3}{t^2 - 6t + 9}$

4. $\lim_{x \to 2} \dfrac{|x - 2|}{x - 2}$

5. $\lim_{x \to 0} \dfrac{\tan x}{x}$

6. $\lim_{x \to 0} \dfrac{\sin 4x}{x}$

Summary

If direct substitution produces a meaningless fractional form 0/0 (an indeterminate form), you must rewrite the fraction so that a new denominator does not have zero as its limit. The limit may exist. If so, the function has a hole.

When direct substitution of c results in a number other than zero over zero, f(x) has a vertical asymptote at x = c, and the left or right hand limit is approaching $+\infty$ or $-\infty$.

Practice

Evaluate the following limits, if they exist.

1. $\lim_{x\to 2}\left(5x^2-3x+1\right)$

2. $\lim_{h\to 0}\left[x\cos(2x)\right]$

3. $\lim_{t\to 3}\left(\frac{x^2-9}{x-3}\right)$

4. $\lim_{x\to 0}\frac{(3+x)^2-3^2}{x}$