Section 4.4 Limits at infinity

What can happen to f(x) as x approaches infinity or negative infinity?

- f(x) approaches a specific value (a horizontal asymptote)
- f(x) approaches infinity or negative infinity (the y-values increase or decrease without bound)
- f(x) is bound by two values (the function oscillates between two values)

Definition:

If $\lim_{x \to \infty} f(x) = b \; or \; \lim_{x \to -\infty} f(x) = b$,
then y = b is a horizontal asymptote of f(x).

Example: $\lim_{x \to \infty} \frac{1}{x}$

Direct substitution: $\frac{1}{\infty} \to 0$

- The numerator is constant and the denominator is increasing without bound, so the overall value is going to zero, therefore the limit is approaching zero.
- A horizontal asymptote of $f(x)=1/x$ is $y = 0$.

Remember

- $\lim_{x \to \infty} f(x) = \frac{\pm\infty}{\#} = \pm\infty$ $\lim_{x \to \infty} f(x) = \frac{\#}{\pm\infty} = 0$
- $\lim_{x \to \infty} f(x) = \frac{\pm\infty}{\pm\infty} = ?$

Example

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

- By direct substitution, numerator and denominator both approach infinity, but its not obvious what happens to the ratio.

- To evaluate the limit at infinity of any rational function, start by dividing the numerator and denominator by the highest power of x that occurs in the denominator.

$$\lim_{x \to \infty} \frac{(3x^2 - x - 2)/x^2}{(5x^2 + 4x + 1)/x^2}$$

$$\lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} - \frac{1}{x^2}} = \frac{3 - 0 - 0}{5 + 0 - 0} = \frac{3}{5}$$

- This is the algebraic reasoning behind the “BOSTON” shortcut for finding horizontal asymptotes.

What is the horizontal asymptote of $f(x) = \frac{2x^2}{x^3+8}$?

- If it is a multiple choice question, do in your head, y = 0.
- If it is a free response question, y = 0 because $\lim_{x\to\infty} \frac{2x^2}{x^3+8} = \lim_{x\to-\infty} \frac{2x^2}{x^3+8} = 0.$ Give a calculus justification involving limits, not a pre-calculus shortcut.

- In pre-calculus you would state the following functions do not have an asymptote. In calculus, we do more than just say it does not have an asymptote. We use limits to describe end behavior.

$$\lim_{x \to \infty} \frac{2x^3 + 5}{3x^2 + 1}$$

$$\lim_{x \to -\infty} \frac{2x^3 + 5}{3x^2 + 1}$$

$$\lim_{x \to \infty} \frac{x^2 + x}{3 - x}$$

- Functions that are not rational may approach different horizontal asymptotes to the left and to the right. Be sure to check in both directions.

- Find the horizontal asymptotes of

$$f(x) = \frac{1}{1+e^{-x}} \qquad y = \frac{3+4^x}{1-4^x}$$