Logistic Growth Model

We have used the exponential growth equation $y = y_0 e^{kt}$ to represent population growth.

The exponential growth equation occurs when the rate of growth is proportional to the amount present.

If we use P to represent the population, the differential equation becomes:

$$\frac{dP}{dt} = kP$$

The constant k is called the **relative** growth rate.

$$\frac{dP/dt}{P} = k$$

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The population growth model becomes: $P = P_0 e^{kt}$

However, real-life populations do not increase forever. There is some limiting factor such as food, living space or waste disposal.

There is a maximum population, or carrying capacity, L.

A more realistic model is the logistic growth model where growth rate is proportional to both the amount present (P) and the carrying capacity that remains: (L-P)

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The equation then becomes:

Logistics Differential Equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right) \quad \text{or} \quad \frac{dP}{dt} = \frac{kP}{L}\left(L - P\right)$$

We can solve this differential equation to find the logistics growth model.

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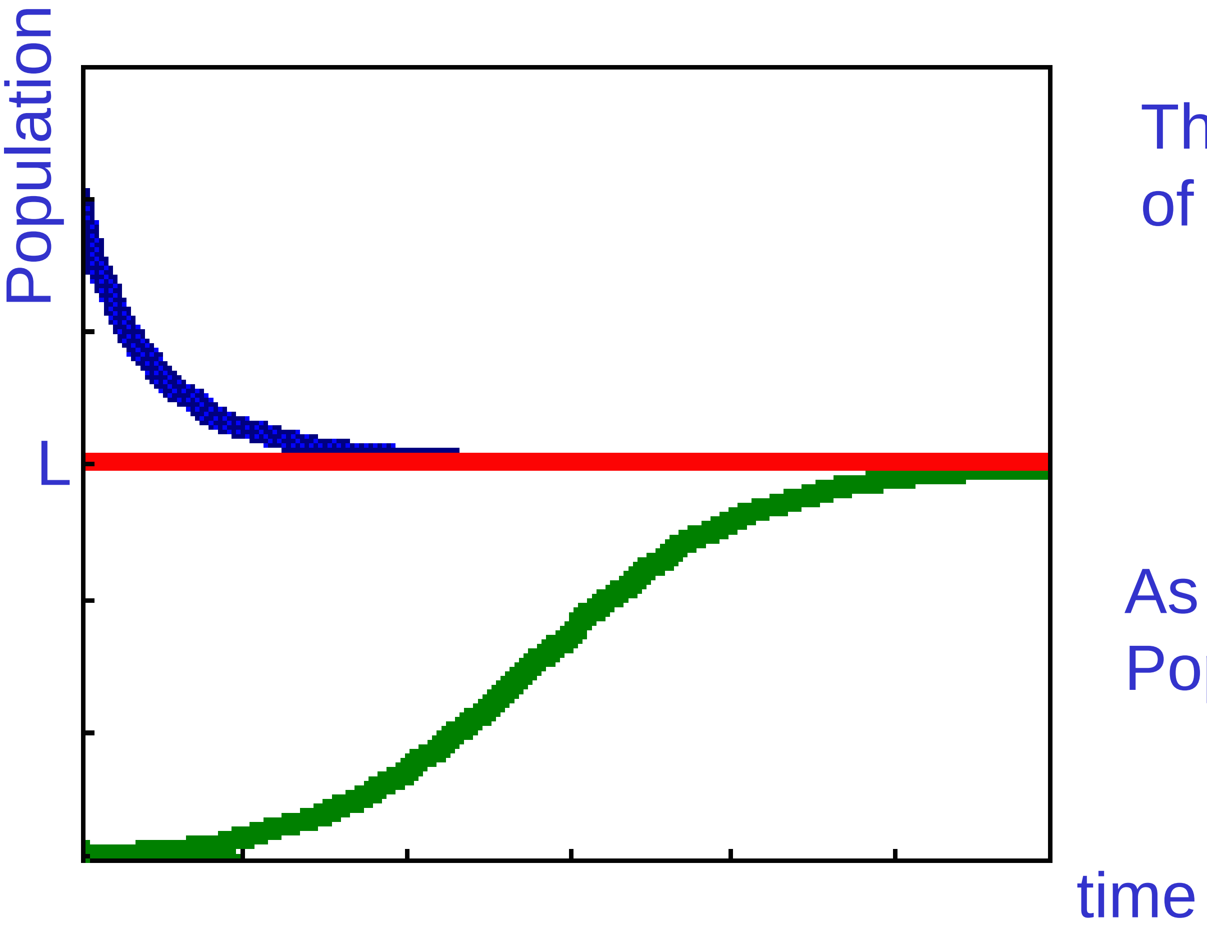
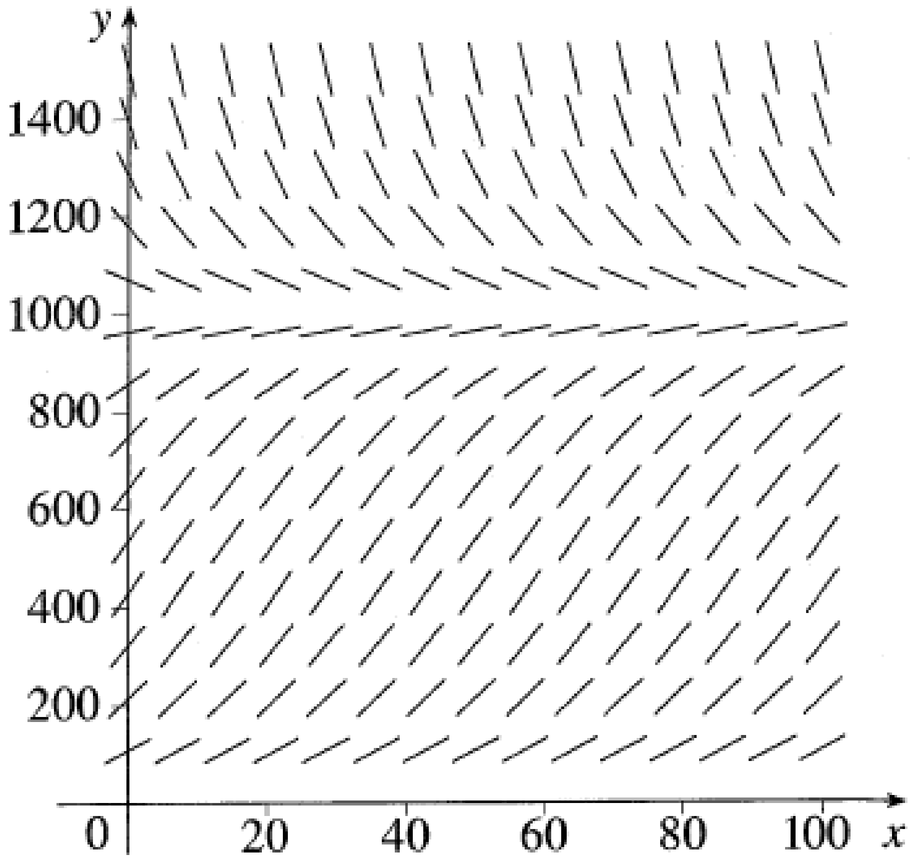


This is an example of a Logistic Curve

As $t \to \infty$,
Population $\to$ L

If $0 < P < L$, $dP/dt > 0$ Population increases

If $P > L$, $dP/dt < 0$ Population decreases



$P' = 0.08P(1 - P/1000)$

This is a slope field for a logistic growth model. What is the carrying capacity? 1000

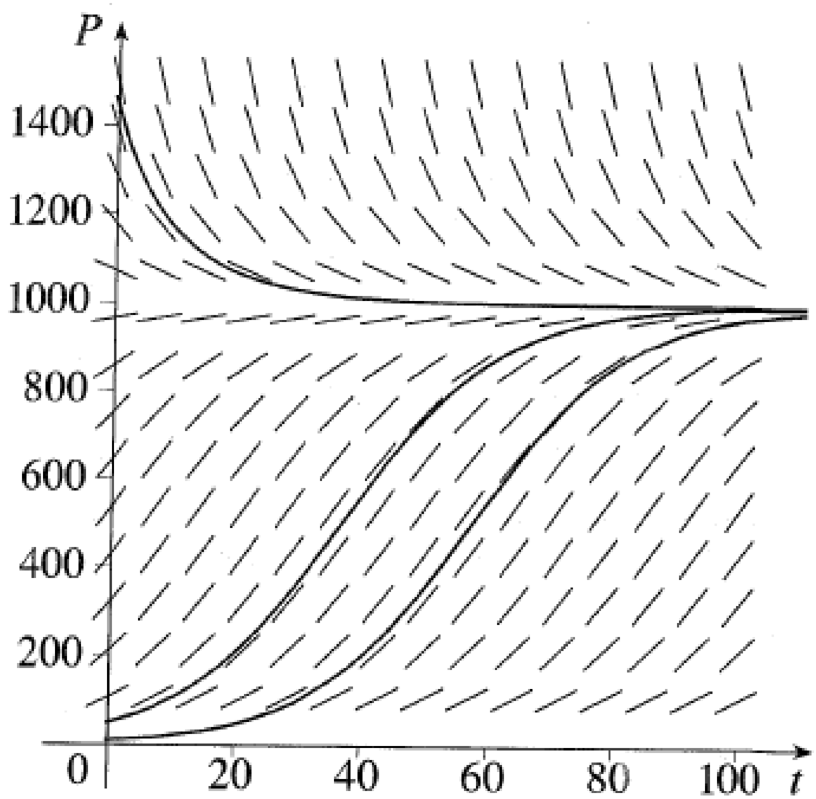
You should be able to pick the carrying capacity out of the equation.

If P(0) = 400

$$\lim_{t\to\infty} P(t) = ?$$ 1000

If P(0) = 1500

$$\lim_{t\to\infty} P(t) = ?$$ 1000



$P' = 0.08P(1 - P/1000)$

Logistic models will always approach the carrying capacity.

The inflection point is when the population is growing the fastest (slope is the greatest).

Let’s find the inflection point.

$$\frac{dP}{dt} = .08P\left(1 - \frac{P}{1000}\right)$$

$$\frac{dP}{dt} = .08P - \frac{.08P^2}{1000}$$

$$\frac{d^2P}{dt^2} = .08P' - \frac{.16PP'}{1000} = 0$$

$$.08P' = \frac{.16PP'}{1000}$$

$$.08 = \frac{.16P}{1000}$$

Inflection point occurs when the 2nd derivative equals zero.

$$P = 500$$

Notice the inflection occurs when the population reaches half of the carrying capacity.

This will always be true.

You can solve the differential equation using Partial Factions.

$$\frac{dP}{dt} = .08P\left(1 - \frac{P}{1000}\right) = \frac{.08}{1000}P(1000 - P)$$

$$P = \frac{1000}{Ae^{-.08t} + 1} \qquad A = \frac{1000 - P_0}{P_0}$$

Logistics Growth Model

$$P = \frac{L}{Ae^{-kt} + 1} \qquad A = \frac{L - P_0}{P_0}$$

This is not something you need to remember.

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Logistics Differential Equation

$$\frac{dP}{dt} = kP(M-P)$$

$$\frac{1}{P(M-P)}dP = kdt$$

$$\frac{1}{\cancel{M}}\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = Mkdt$$

$$\ln P - \ln(M-P) = Mkt + C$$

$$\ln\frac{P}{M-P} = Mkt + C$$

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

$$1 = A(M-P) + BP$$

$\boxed{\text{Partial Fractions}}$

$$1 = AM - AP + BP$$

$$1 = AM \qquad 0 = -AP + BP$$

$$AP = BP$$

$$\frac{1}{M} = A \qquad A = B$$

$$\frac{1}{M} = B$$

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Logistics Differential Equation

$$\frac{dP}{dt} = kP\left(M - P\right)$$

$$\frac{1}{P\left(M-P\right)}dP = kdt$$

$$\frac{1}{\cancel{M}}\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = Mkdt$$

$$\ln P - \ln\left(M - P\right) = Mkt + C$$

$$\ln \frac{P}{M-P} = Mkt + C$$

$$\frac{P}{M-P} = e^{Mkt+C}$$

$$\frac{M-P}{P} = e^{-Mkt-C}$$

$$\frac{M}{P} - 1 = e^{-Mkt-C}$$

$$\frac{M}{P} = 1 + e^{-Mkt-C}$$

Logistics Differential Equation

$$\frac{P}{M-P}=e^{Mkt+C}$$

$$\frac{M-P}{P}=e^{-Mkt-C}$$

$$\frac{M}{P}-1=e^{-Mkt-C}$$

$$\frac{M}{P}=1+e^{-Mkt-C}$$

$$P=\frac{M}{1+e^{-Mkt-C}}$$

$$P=\frac{M}{1+e^{-C}\cdot e^{-Mkt}}$$

$$Let\ \ A=e^{-C}$$

$$\boxed{P=\frac{M}{1+Ae^{-(Mk)t}}}$$

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