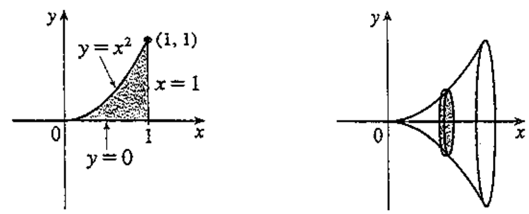
## 6.2 Volumes

**1.** A cross-section is circular with radius $x^2$, so its area is $A(x) = \pi(x^2)^2$.

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2)^2\,dx = \pi\int_0^1 x^4\,dx = \pi\left[\tfrac{1}{5}x^5\right]_0^1 = \frac{\pi}{5}$$

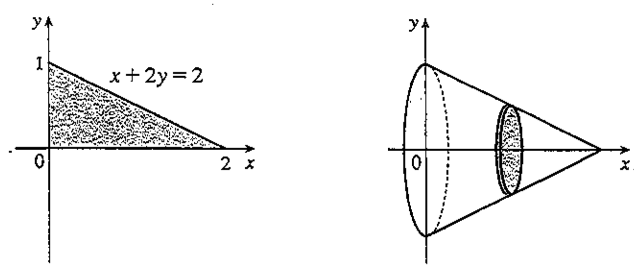
$2\pi$




**2.** $x + 2y = 2 \iff y = 1 - \frac{1}{2}x$, so a cross-section is circular with radius $1 - \frac{1}{2}x$, and its area is $A(x) = \pi\left(1 - \frac{1}{2}x\right)^2$.

$$V = \int_0^2 \pi y^2\,dx = \pi\int_0^2 \left(1 - \tfrac{1}{2}x\right)^2\,dx = \pi\int_0^2 \left(1 - x + \tfrac{1}{4}x^2\right)dx = \pi\left[x - \tfrac{1}{2}x^2 + \tfrac{1}{12}x^3\right]_0^2$$
$$= \pi\left(2 - 2 + \tfrac{2}{3}\right) = \tfrac{2}{3}\pi$$

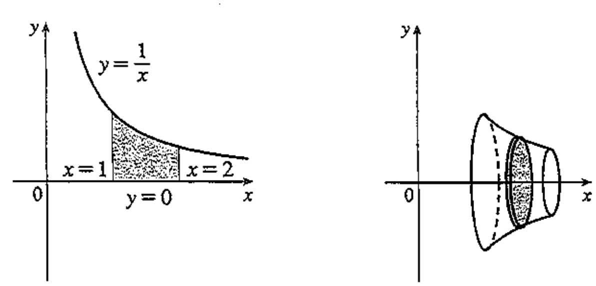
$\frac{2}{3}\pi$




**3.** A cross-section is a disk with radius $1/x$, so its area is $A(x) = \pi(1/x)^2$.

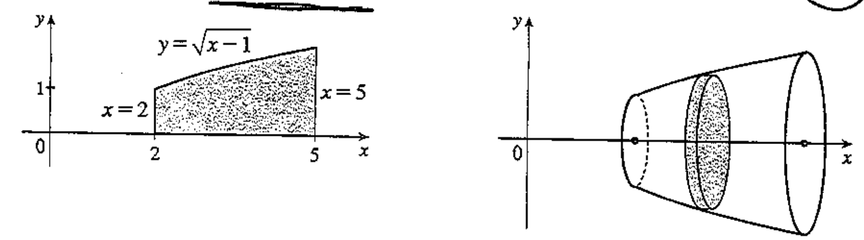
$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 = \pi\left[-\tfrac{1}{2} - (-1\right.$$

$\frac{\pi}{2}$



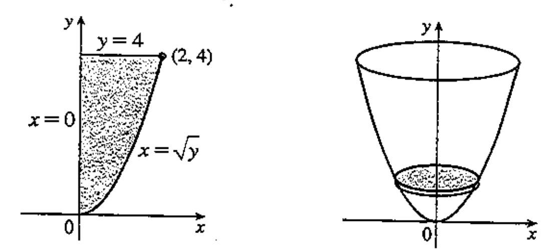

**4.** A cross-section is circular with radius $\sqrt{x-1}$, so its area is $A(x) = \pi\left(\sqrt{x-1}\right)^2 = \pi(x-1)$.

$$V = \int_2^5 A(x)\,dx = \int_2^5 \pi(x-1)\,dx = \pi\left[\tfrac{1}{2}x^2 - x\right]_2^5 = \pi\left(\tfrac{25}{2} - 5 - \tfrac{4}{2} + 2\right) = \tfrac{15}{2}\pi$$



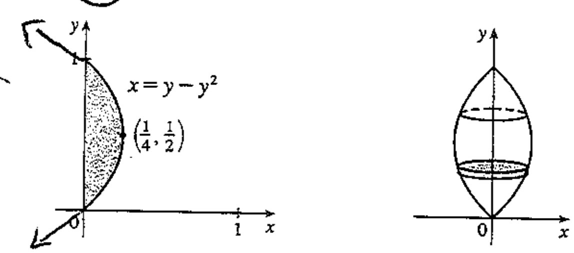

**5.** A cross-section is a disk with radius $\sqrt{y}$, so its area is $A(y) = \pi\left(\sqrt{y}\right)^2$.

$$V = \int_0^4 A(y)\,dy = \int_0^4 \pi\left(\sqrt{y}\right)^2 dy = \pi\int_0^4 y\,dy = \pi\left[\tfrac{1}{2}y^2\right]_0^4 = 8\pi$$



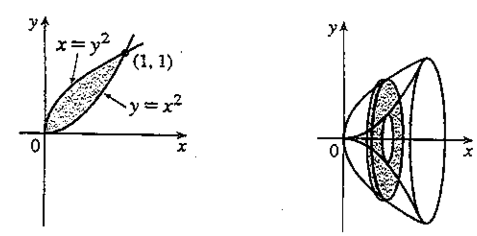

**6.** A cross-section is a disk with radius $y - y^2$, so its area is $A(y) = \pi\left(y - y^2\right)^2$.

$$V = \int_0^1 A(y)\,dy = \int_0^1 \pi\left(y - y^2\right)^2 dy = \pi\int_0^1 \left(y^4 - 2y^3 + y^2\right)dy = \pi\left[\tfrac{1}{5}y^5 - \tfrac{1}{2}y^4 + \tfrac{1}{3}y^3\right]_0^1$$
$$= \pi\left(\tfrac{1}{5} - \tfrac{1}{2} + \tfrac{1}{3}\right) = \frac{\pi}{30}$$




**7.** A cross-section is a washer (annulus) with inner radius $x^2$ and outer radius $\sqrt{x}$, so its area is $A(x) = \pi\left(\sqrt{x}\right)^2 - \pi\left(x^2\right)^2 = \pi\left(x - x^4\right)$.

$$V = \int_0^1 A(x)\,dx = \pi\int_0^1 \left(x - x^4\right)dx = \pi\left[\tfrac{1}{2}x^2 - \tfrac{1}{5}x^5\right]_0^1 = \pi\left(\tfrac{1}{2} - \tfrac{1}{5}\right) = \tfrac{3\pi}{10}$$

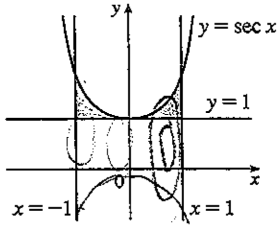


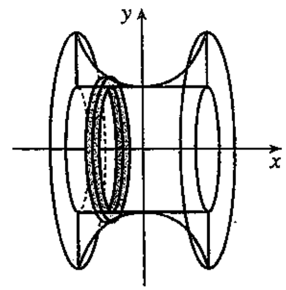
**8.** A cross-section is a washer with inner radius 1 and outer radius $\sec x$, so its area is

$A(x) = \pi(\sec x)^2 - \pi(1)^2 = \pi(\sec^2 x - 1)$.

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(\sec^2 x - 1)\,dx = 2\pi \int_0^1 (\sec^2 x - 1)\,dx = 2\pi[\tan x - x]_0^1 = 2\pi(\tan 1 - 1)$$

$\approx 3.5023$

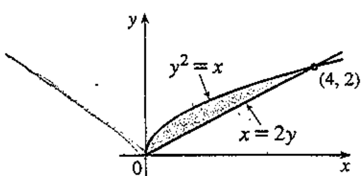


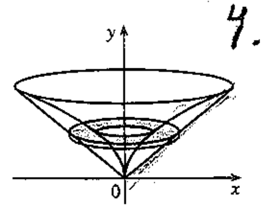




**9.** A cross-section is a washer with inner radius $y^2$ and outer radius $2y$, so its area is

$A(y) = \pi(2y)^2 - \pi(y^2)^2 = \pi(4y^2 - y^4)$.

$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 (4y^2 - y^4)\,dy = \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64\pi}{15}$$



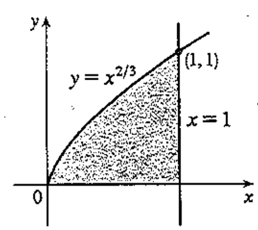



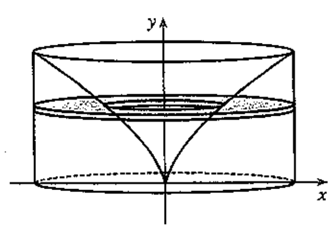

4. 267π

**10.** $y = x^{2/3} \Leftrightarrow x = y^{3/2}$, so a cross-section is a washer with inner radius $y^{3/2}$ and outer radius 1,

$A(y) = \pi(1)^2 - \pi\left(y^{3/2}\right)^2 = \pi(1 - y^3)$.

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 (1 - y^3)\,dy = \pi\left[y - \tfrac{1}{4}y^4\right]_0^1 = \tfrac{3}{4}\pi$$



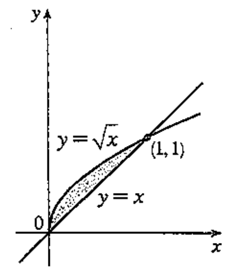

**11.** A cross-section is a washer with inner radius $1 - \sqrt{x}$ and outer radius $1 - x$, so its area is

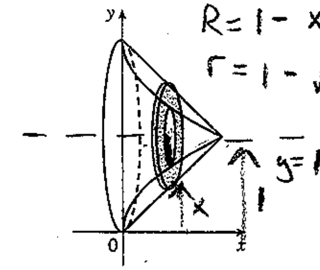
$A(x) = \pi(1 - x)^2 - \pi(1 - \sqrt{x})^2 = \pi[(1 - 2x + x^2) - (1 - 2\sqrt{x} + x)] = \pi(-3x + x^2 + 2\sqrt{x})$.

$$V = \int_0^1 A(x)\,dx = \pi \int_0^1 (-3x + x^2 + 2\sqrt{x})\,dx$$

$$= \pi\left[-\tfrac{3}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{4}{3}x^{3/2}\right]_0^1 = \pi\left(-\tfrac{3}{2} + \tfrac{5}{3}\right) = \tfrac{\pi}{6}$$

top-bottom



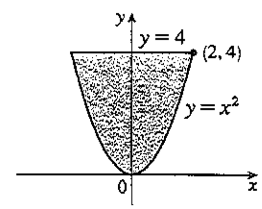

R = 1 - x

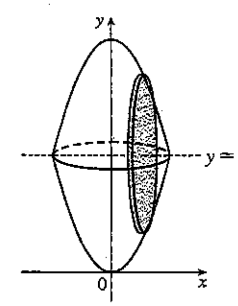
r = 1 - √x

**12.** A cross-section is circular with radius $4 - x^2$, so its area is $A(x) = \pi(4 - x^2)^2 = \pi(16 - 8x^2 + x^4)$.

$$V = \int_{-2}^{2} A(x)\,dx = 2\int_0^2 A(x)\,dx = 2\pi \int_0^2 (16 - 8x^2 + x^4)\,dx = 2\pi\left[16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(32 - \tfrac{64}{3} + \tfrac{32}{5}\right) = 64\pi\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = 64\pi \cdot \tfrac{8}{15} = \tfrac{512\pi}{15}$$ = 34. 133 π



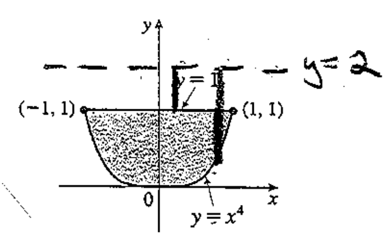

**13.** A cross-section is an annulus with inner radius $2 - 1$ and outer radius $2 - x^4$, so its area is

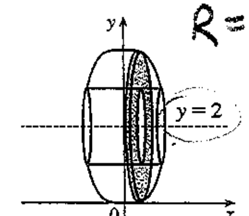
$A(x) = \pi(2 - x^4)^2 - \pi(2 - 1)^2 = \pi(3 - 4x^4 + x^8)$.

$$V = \int_{-1}^{1} A(x)\,dx = 2\int_0^1 A(x)\,dx = 2\pi \int_0^1 (3 - 4x^4 + x^8)\,dx = 2\pi\left[3x - \tfrac{4}{5}x^5 + \tfrac{1}{9}x^9\right]_0^1$$

$$= 2\pi\left(3 - \tfrac{4}{5} + \tfrac{1}{9}\right) = \tfrac{208}{45}\pi$$ 4.622π

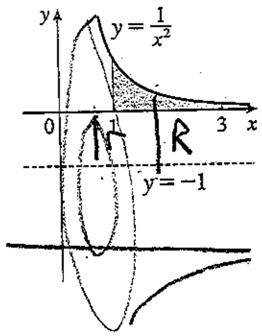
R = top-bottom

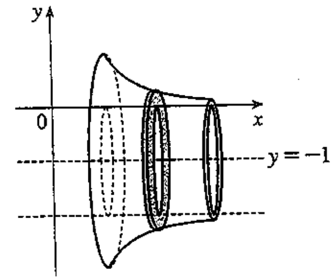


**14.** $V = \int_1^3 \pi \left\{ \left[ \frac{1}{x^2} - (-1) \right]^2 - [0-(-1)]^2 \right\} dx = \pi \int_1^3 \left[ \left( \frac{1}{x^2} + 1 \right)^2 - 1^2 \right] dx = \pi \int_1^3 \left( \frac{1}{x^4} + \frac{2}{x^2} \right) dx$

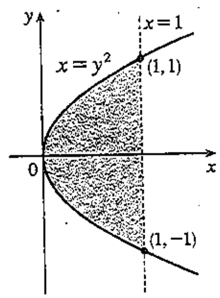
$= \pi \left[ -\frac{1}{3x^3} - \frac{2}{x} \right]_1^3 = \pi [ (-\frac{1}{81} - \frac{2}{3}) - (-\frac{1}{3} - 2) ] = \frac{134\pi}{81}$

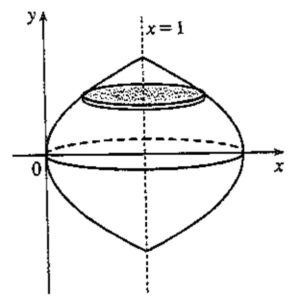



**15.** $V = \int_{-1}^1 \pi (1-y^2)^2 \, dy = 2 \int_0^1 \pi (1-y^2)^2 \, dy = 2\pi \int_0^1 (1 - 2y^2 + y^4) \, dy$

$= 2\pi \left[ y - \frac{2}{3}y^3 + \frac{1}{5}y^5 \right]_0^1 = 2\pi \cdot \frac{8}{15} = \frac{16}{15}\pi$ $= 1.067\pi$



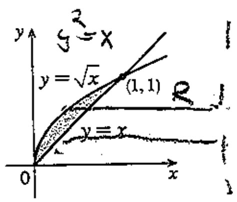



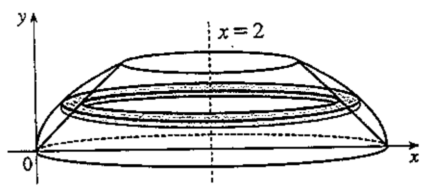

**16.** $y = \sqrt{x} \;\; \Rightarrow \;\; x = y^2$, so the outer radius is $2 - y^2$.

$V = \int_0^1 \pi \left[ (2-y^2)^2 - (2-y)^2 \right] dy = \pi \int_0^1 \left[ (4 - 4y^2 + y^4) - (4 - 4y + y^2) \right] dy$

$= \pi \int_0^1 (y^4 - 5y^2 + 4y) \, dy = \pi \left[ \frac{1}{5}y^5 - \frac{5}{3}y^3 + 2y^2 \right]_0^1 = \pi \left( \frac{1}{5} - \frac{5}{3} + 2 \right) = \frac{8}{15}\pi$

$.533\pi$



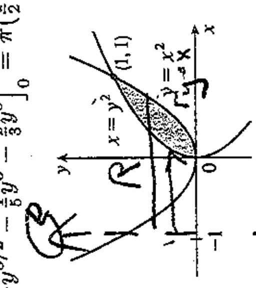



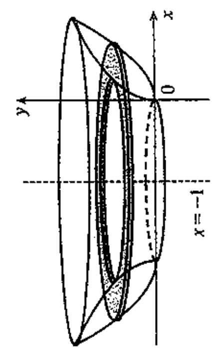

**17.** $y = x^2 \;\; \Rightarrow \;\; x = \sqrt{y}$ for $x \geq 0$. The outer radius is the distance from $x = -1$ to $x = \sqrt{y}$ and the inner radius is the distance from $x = -1$ to $x = y^2$.

$V = \int_0^1 \pi \left\{ \left[ \sqrt{y} - (-1) \right]^2 - \left[ y^2 - (-1) \right]^2 \right\} dy = \pi \int_0^1 \left[ (\sqrt{y} + 1)^2 - (y^2 + 1)^2 \right] dy$

$= \pi \int_0^1 (y + 2\sqrt{y} + 1 - y^4 - 2y^2 - 1) \, dy = \pi \int_0^1 (y + 2\sqrt{y} - y^4 - 2y^2) \, dy$

$= \pi \left[ \frac{1}{2}y^2 + \frac{4}{3}y^{3/2} - \frac{1}{5}y^5 - \frac{2}{3}y^3 \right]_0^1 = \pi \left( \frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3} \right) = \frac{29}{30}\pi$ $\quad \frac{29}{30}\pi = .967\pi$




**18.** For $0 \leq y < 2$, a cross-section is an annulus with inner radius $2 - 1$ and outer radius $4 - 1$, the area of which is $A_1(y) = \pi(4-1)^2 - \pi(2-1)^2$. For $2 \leq y \leq 4$, a cross-section is an annulus with inner radius $y - 1$ and outer radius $4 - 1$, the area of which is $A_2(y) = \pi(4-1)^2 - \pi(y-1)^2$.

$V = \int_0^4 A(y) \, dy = \pi \int_0^2 \left[ (4-1)^2 - (2-1)^2 \right] dy + \pi \int_2^4 \left[ (4-1)^2 - (y-1)^2 \right] dy$

$= \pi [8y]_0^2 + \pi \int_2^4 (8 + 2y - y^2) \, dy = 16\pi + \pi \left[ 8y + y^2 - \frac{1}{3}y^3 \right]_2^4$

$= 16\pi + \pi \left[ (32 + 16 - \frac{64}{3}) - (16 + 4 - \frac{8}{3}) \right] = \frac{76}{3}\pi$ $\quad 25.33\pi$