Sections 9.2 and 9.3 Parametric Equations

Graphing, Eliminating the Parameter, and finding the first derivative

Types of Functions

Single-Variable

Multi-Variable

Parametric

Single Variable Functions

Until now, all the functions we have studied have had one independent and one dependent variable.

$$x \longrightarrow \boxed{\quad\quad} \longrightarrow f(x)$$

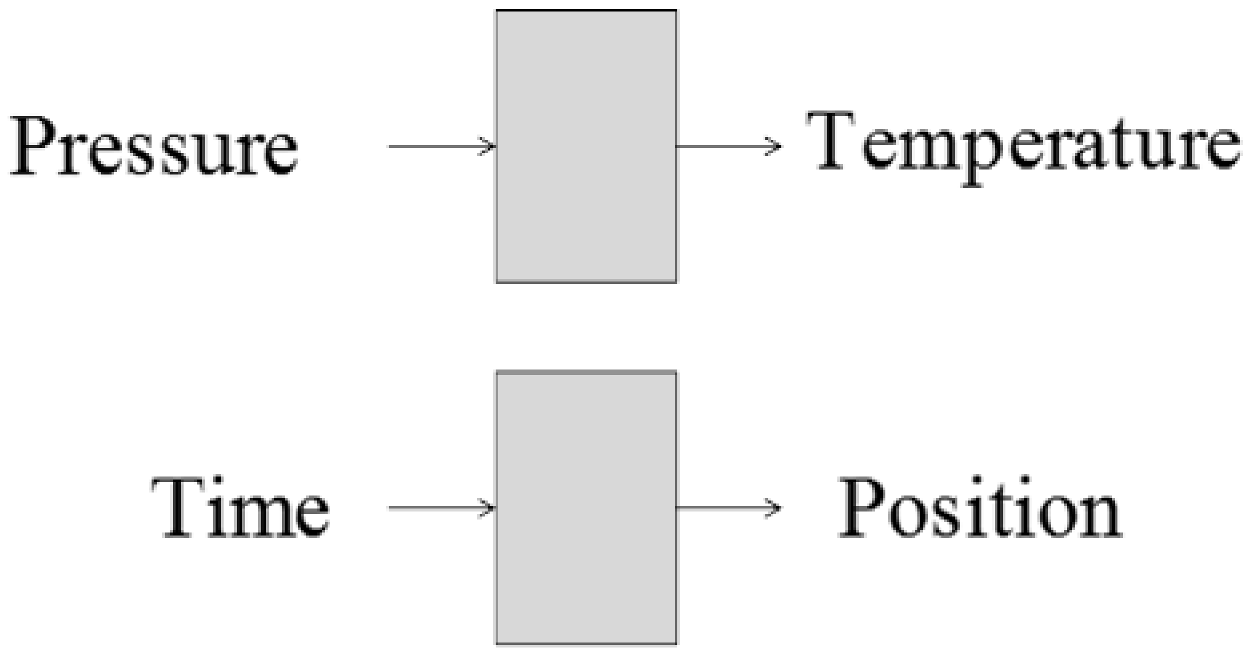
Independent
Variables you control

Dependent
What results

$$f(x) = x^2$$

The Calculus of these types of functions is known as Single-Variable Calculus because the functions have a single independent variable.

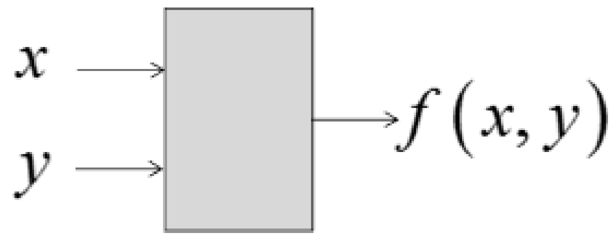
Examples of Single-variable Functions



$$f(x) = x^2$$

Multivariable Functions

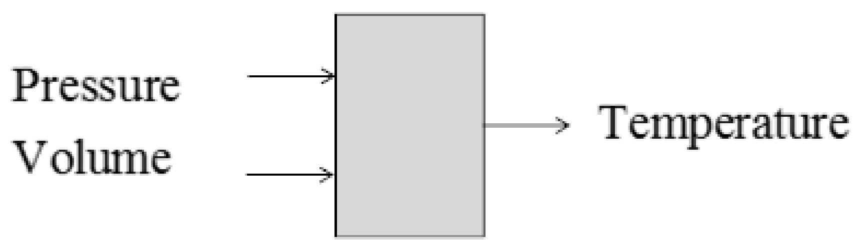
But you can have more than one independent variable.



$$f(x,y) = x^2 + y^2$$

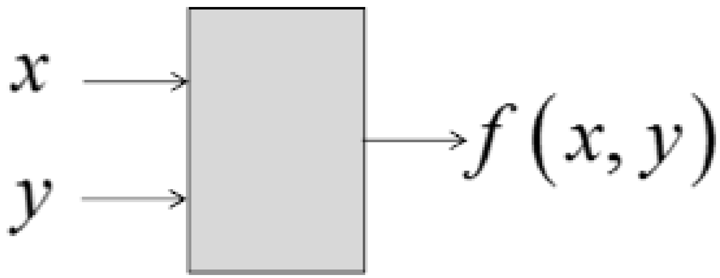
The Calculus of these types of functions is known as Multi-Variable Calculus because the functions have more than one independent variable.
You will study these in Calculus III.

Example:



Which brings us to Parametrics

Multivariable

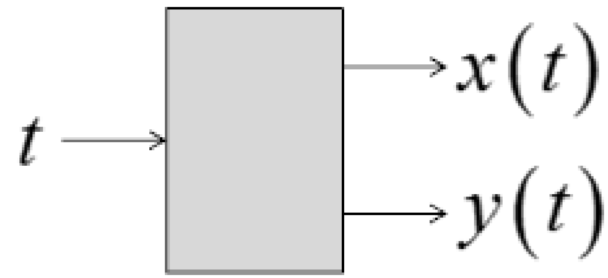


$$f(x,y)=x^2+y^2$$

Multiple independent variables

One function value

Parametric



$$x=t^2+\cos t$$

$$y=t^2+\sin t$$

One independent variable (the “parameter”)

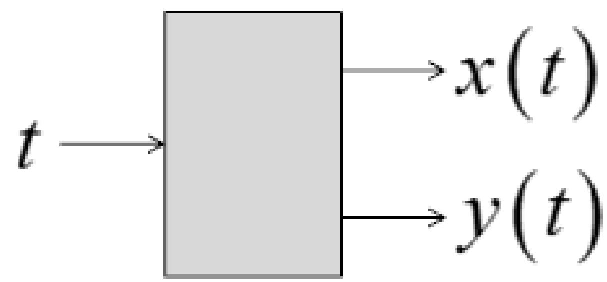
Multiple function values

Parametric Example

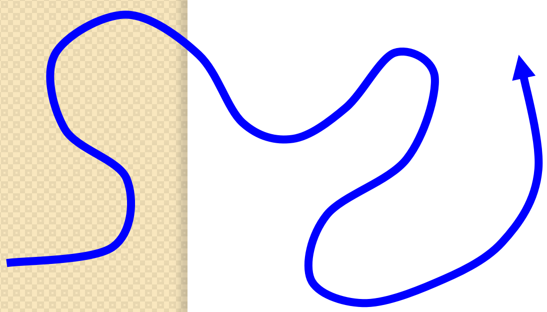
t is in years

$x(t) = 4\cos t + 5$ represents the population of field mice

$y(t) = 2\sin t + 5$ represents the population of hawks



There are times when we need to describe motion (or a curve) that is *not a function*.



We can do this by writing equations for the x and y coordinates in terms of a third variable (usually *time* or θ).

$$x = f(t) \quad y = g(t)$$

Each parametric equation gives one of the coordinates of position at a certain time.

Parametric Equations can be used to answer questions of where and when.

Example: Write a set of parametric equations for the line $y = \frac{1}{2}x + 3$

One possible answer

$$A: \begin{cases} x = t \\ y = \frac{1}{2}t + 3 \end{cases}$$

Another possible answer

$$B: \begin{cases} x = 2t \\ y = \frac{1}{2}(2t) + 3 = t + 3 \end{cases}$$

Graphs look the same, but parametric equations show location at a certain time.

When t = 4, Graph A is at (4, 5) and Graph B is at (8, 7)

Both are traveling along the same line, just B is traveling faster

View the equations on your calculator.

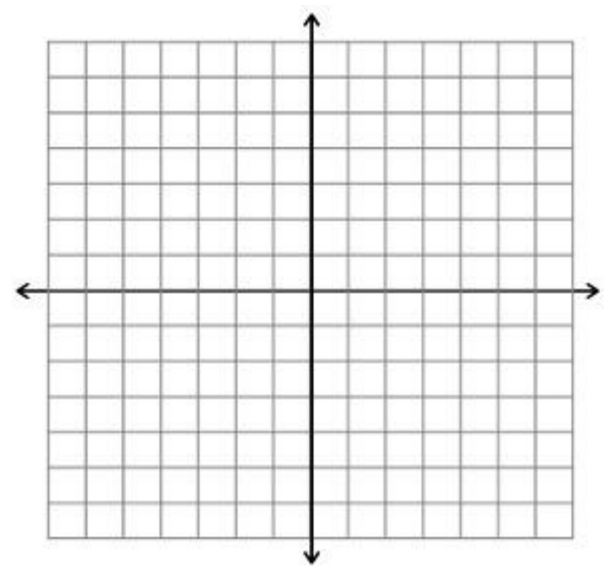
Example:

Make a table of values and sketch the curve, indicating the direction of the curve. Then eliminate the parameter.

$$x = \sqrt{t}, \quad y = t + 1$$

We can only use nonnegative values for t.

t	x	y



To eliminate the parameter, solve one equation for t, and substitute into the other.

$$x = \sqrt{t} \qquad y = t + 1$$

$$x^2 = t \qquad y = x^2 + 1$$

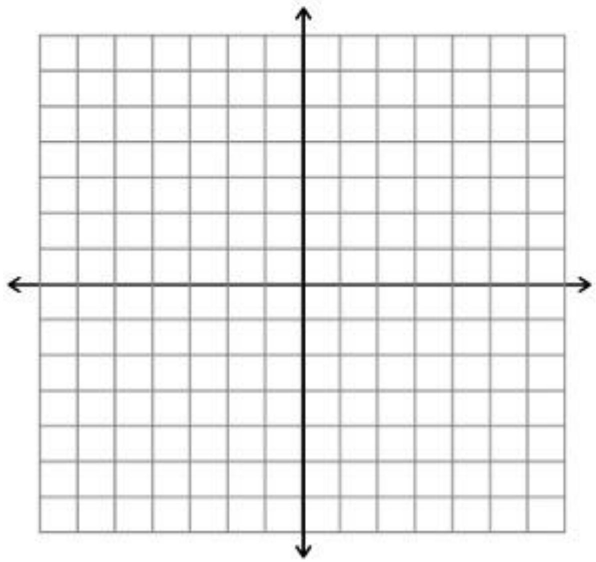
To make the equation match the graph, we must restrict x to be greater than or equal to zero.

$$y = x^2 + 1, x \geq 0$$

Graph the plane curve represented by the parametric equations

$$x = 3 + 2\cos t, \quad y = -1 + 3\sin t; \quad \boxed{0 \le t \le 2\pi}$$

t	x	y	(x, y)
0	5	-1	$(5,-1)$
$\frac{\pi}{2}$	3	2	$(3,2)$
π	1	-1	$(1,-1)$
$\frac{3\pi}{2}$	3	-4	$(3,-4)$
2π	5	-1	$(5,-1)$



Now, eliminate the parameter. Based on our curve we'd expect to get the equation of an ellipse.

$$x = 3 + 2\cos t, \quad y = -1 + 3\sin t; \quad 0 \le t \le 2\pi$$

$$\frac{x-3}{2} = \frac{\cancel{2}\cos t}{\cancel{2}}, \quad \frac{y+1}{3} = \frac{\cancel{3}\sin t}{\cancel{3}}$$

When you want to eliminate the parameter and you have trig functions, it is not easy to solve for *t* because it requires inverse functions. Instead you solve for cos *t* and sin *t* and substitute them in the Pythagorean Identity:

$$\frac{x-3}{2} = \cos t \quad \frac{y+1}{3} = \sin t$$

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{y+1}{3}\right)^2 + \left(\frac{x-3}{2}\right)^2 = 1$$

$$\boxed{\frac{(y+1)^2}{9} + \frac{(x-3)^2}{4} = 1}$$

$$\frac{dy}{dt}$$ = Change in y over time

Tells you if the path is moving up or down

$$\frac{dx}{dt}$$ = Change in x over time

Tells you if the path is moving left or right

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$ = Slope of tangent line to the curve

Given: $\begin{cases} x = 4\sin t \\ y = 2\cos t \end{cases}$ Find dx/dt , dy/dt , and dy/dx at t = π/4

$$\frac{dx}{dt} = 4\cos t \qquad \left.\frac{dx}{dt}\right|_{t=\frac{\pi}{4}} = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

Positive change indicates the path is moving right

$$\frac{dy}{dt} = -2\sin t \qquad \left.\frac{dy}{dt}\right|_{t=\frac{\pi}{4}} = -2\left(\frac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

Negative change indicates the path is moving down

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sin t}{4\cos t} = \frac{-1}{2}\tan t \qquad \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{-\sqrt{2}}{2\sqrt{2}} = \frac{-1}{2}$$

Example: Describe the movement and write an equation of the tangent line when t = 1

$$\begin{cases} x = 2t^2 + t \\ y = t^3 + 2t^2 + t \end{cases}$$

Example: Find all the points of vertical and horizontal tangency.

$$\begin{cases} x = t^2 + t \\ y = t^2 - 3t + 5 \end{cases}$$

Practice:

- p. 650 #1, 3, 9, 15, 17, 23
- p. 657 #17, 21, 23, and describe the movement of the particle, 31, 35, 39

More Parametric Equations

Second Derivatives and Arc Length

Second Derivatives

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$x = \sin t$$
$$y = t^2 + 1$$

Find $\frac{dy}{dx} = \boxed{\frac{2t}{\cos t}}$

To find the <u>second</u> derivative of a parametrized curve, we find the derivative of the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{2t}{\cos t}\right) = \frac{\cos t(2) - 2t(-\sin t)}{\cos^2 t} \cdot \left(\frac{dt}{dx}\right)$$

Quotient rule

Chain rule

$$= \frac{\cos t(2) - 2t(-\sin t)}{\cos^2 t} \div \left(\frac{dx}{dt}\right)$$

Change to division because we know dx/dt, not dt/dx

$$= \frac{\cos t(2) - 2t(-\sin t)}{\cos^2 t} \div \cos t = \boxed{\frac{\cos t(2) - 2t(-\sin t)}{\cos^3 t}}$$

Let's Generalize

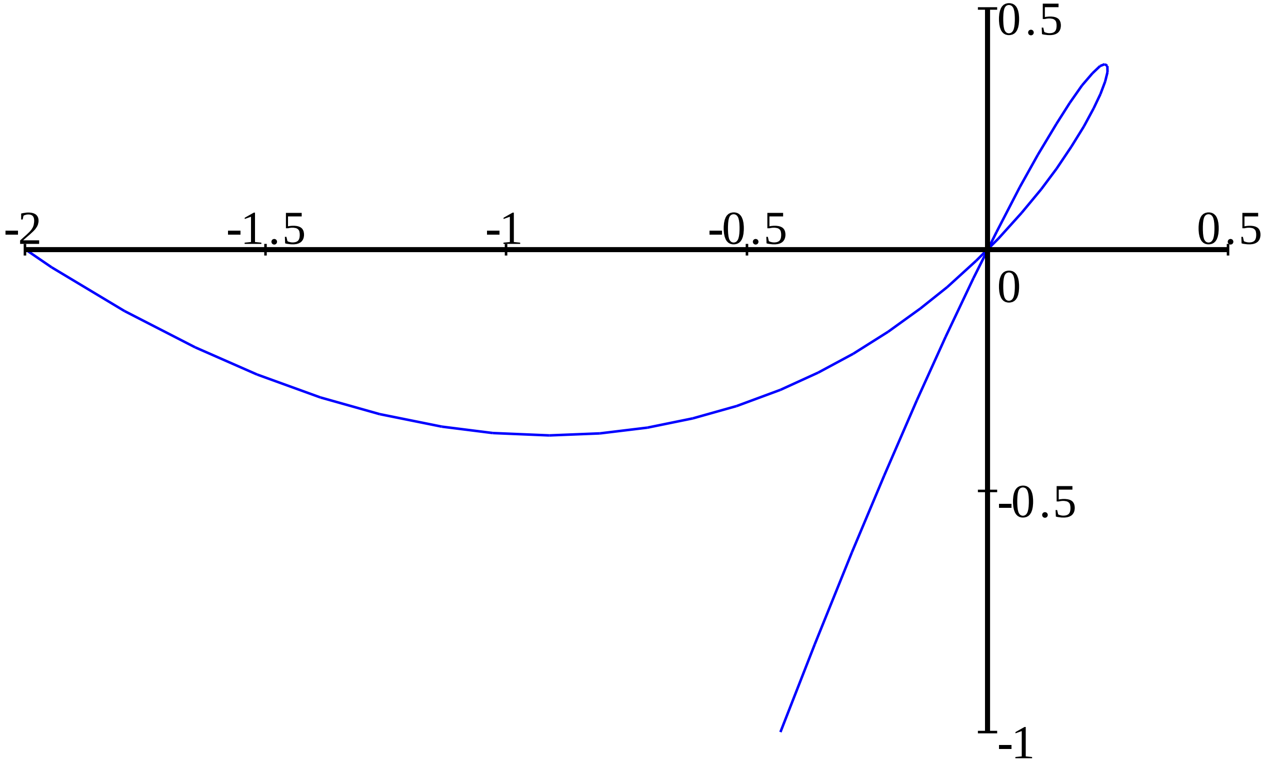
$$\frac{d^2 y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

The derivative of the first derivative using normal derivative rules

Divided by dx/dt <u>again</u> because of the chain rule

Example

Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.



Example (cont.):

Find $\dfrac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

1. Find the first derivative (dy/dx).

$$y' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t}$$

→

2. Find the derivative of *dy*/*dx* with respect to *t*.

$$\frac{dy'}{dt}=\frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right)=\frac{(1-2t)(-6t)-(1-3t^2)(-2)}{(1-2t)^2}$$

$$=\frac{2-6t+6t^2}{(1-2t)^2}$$

→

3. Divide by *dx*/*dt*.

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{\mathrm{dx}}{\mathrm{dt}}} = \frac{\frac{2-6t+6t^2}{(1-2t)^2}}{1-2t}$$

$$= \frac{2-6t+6t^2}{(1-2t)^3}$$

Find the slope and concavity at (2, 3)

$$\begin{cases} x = \sqrt{t} \\ y = \dfrac{1}{4}t^2 - 1 \end{cases}$$

Recall The Formula for Arc Length:

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

$$L = \int_a^b \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx = \int_a^b \sqrt{1 + \left[\frac{dy/dt}{dx/dt}\right]^2}\, dx$$

$$=\int_a^b \sqrt{1+\left[\frac{dy/dt}{dx/dt}\right]^2}\,dx=\int_a^b \sqrt{\frac{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\,dx$$

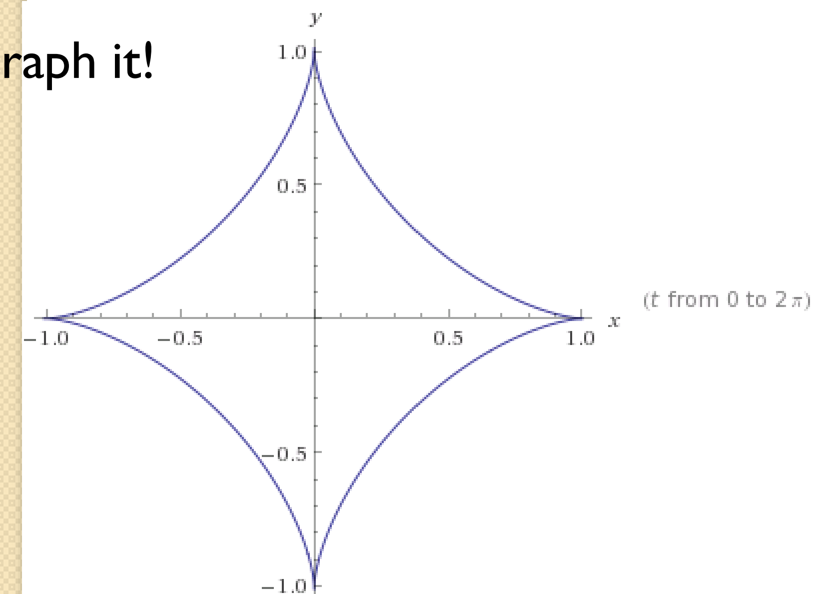
$$=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,\cancel{\frac{dt}{dx}dx}$$

Arc Length of a Parametric Equation

$$L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Find the length of $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}, 0 \le t \le 2\pi$

Graph it!



Can't take the derivative at a cusp.

But the curve is symmetrical, so find the length of one part and multiply by 4.

Practice:

- P. 657 #5-13 odd, 47-51 odd, 88-90