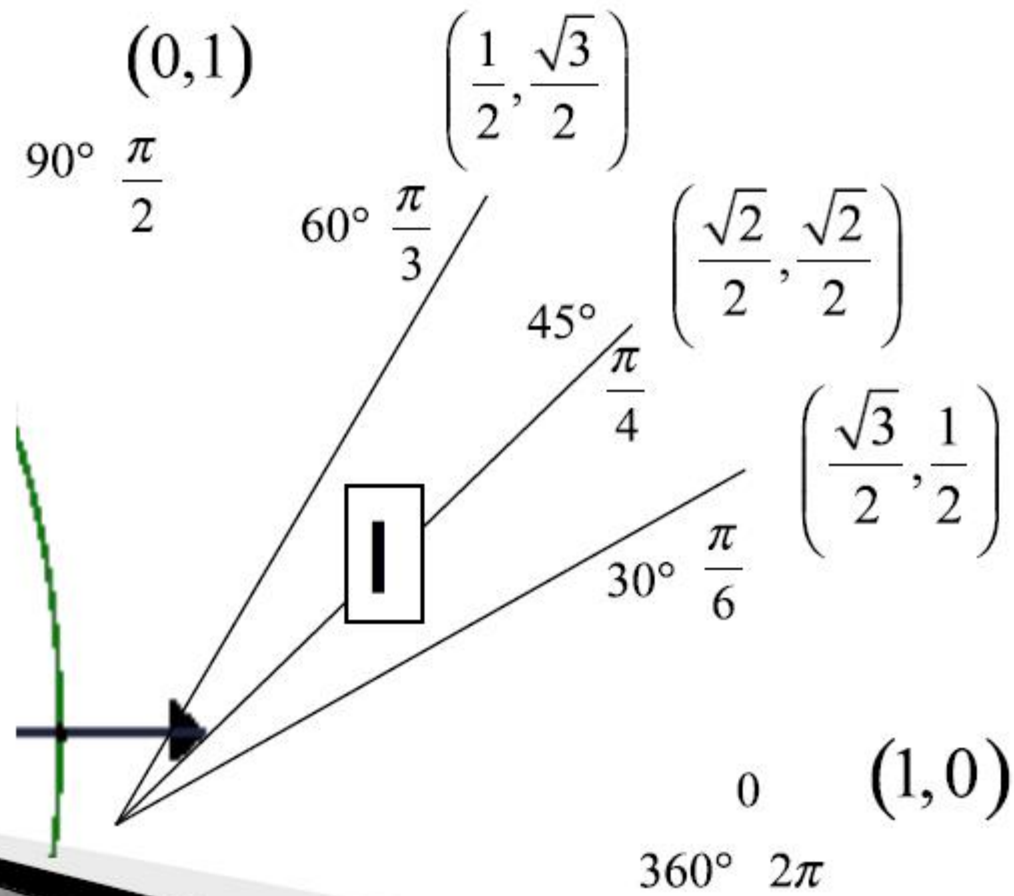
Polar and Parametric

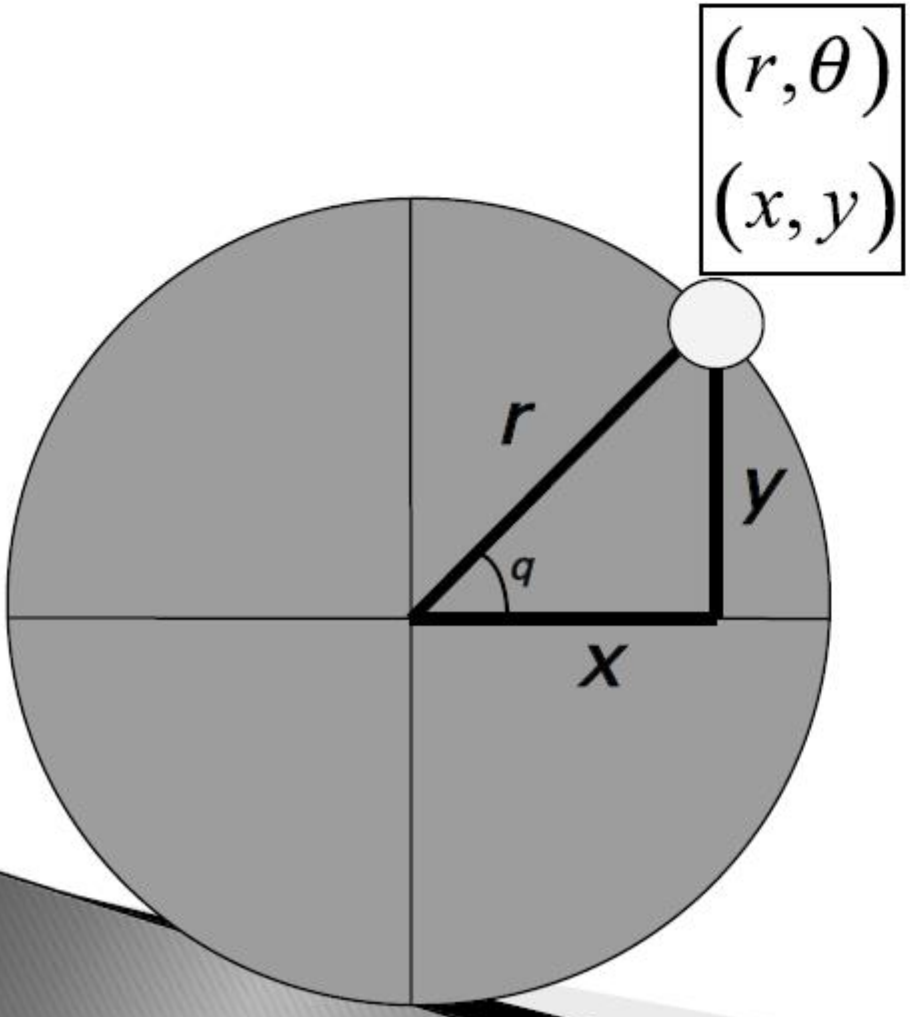
Polar Functions

Know the unit circle!!!

$$(\cos, \sin)$$



Polar to/from Cartesian



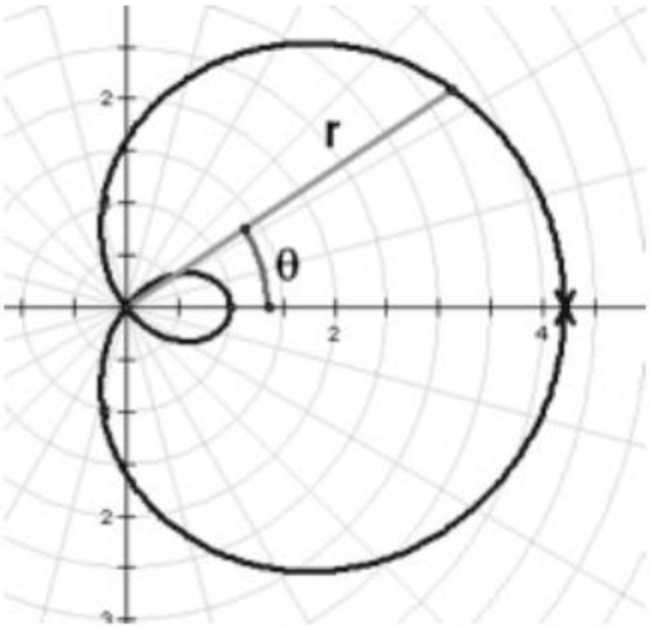
$$r = \sqrt{x^2 + y^2}$$

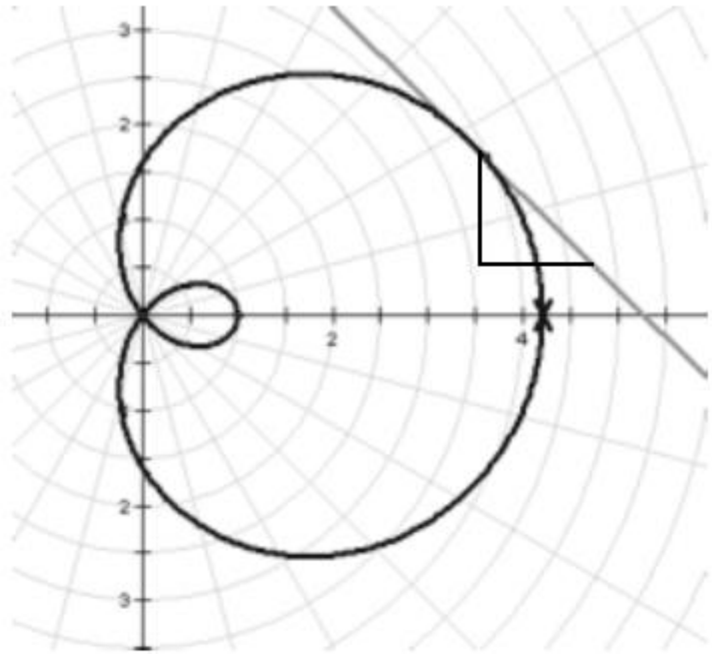
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\cos\theta = \frac{x}{r} \qquad x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \qquad y = r\sin\theta$$

Compare and Contrast





$$\frac{dr}{d\theta} =$$ Rate of change in radius as theta changes.

$$\frac{dy}{dx} =$$ Slope of the tangent

$$y = r\sin\theta$$

$$x = r\cos\theta$$

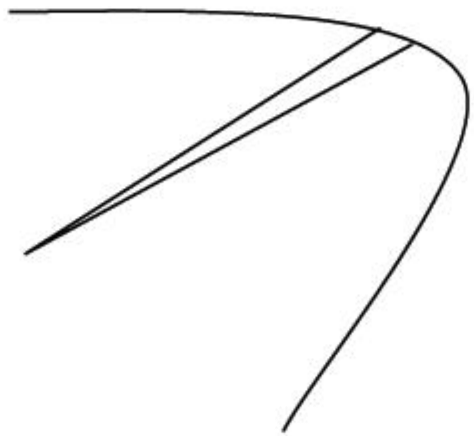
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \boxed{\frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

Product Rule!

And if I wanted horizontal tangents, $\frac{dy}{d\theta} = 0$

If I wanted vertical tangents, $\frac{dx}{d\theta} = 0$

Polar Area



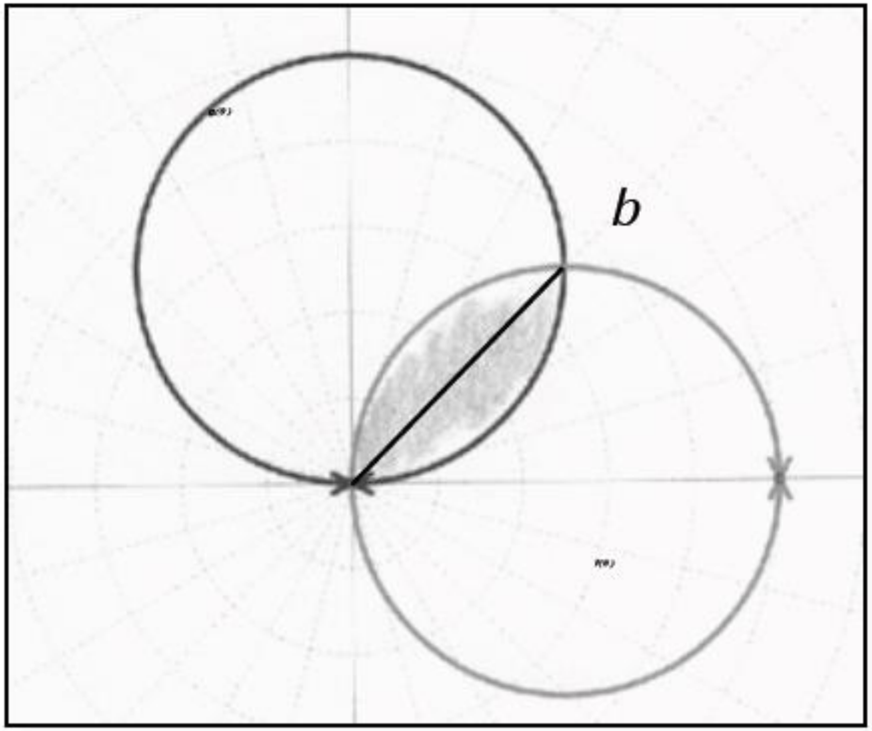
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Just like Riemann sums and rectangles.

Finding the limits…

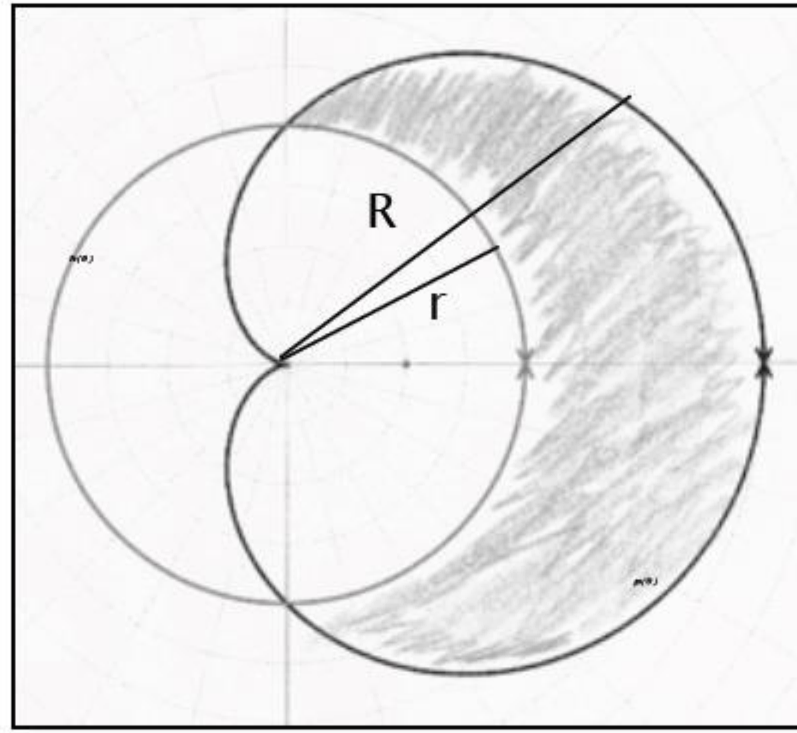
- Bounds (limits) on integrals must be largest interval the does not result in graph tracing over itself.
- Most simple functions are 0 to 2π.
- Rose curves with odd petals are just 0 to π.
- More complicated functions, use calculator trace.
- You can also use symmetry to simplify the integral

Distinguish between piecewise and area between curves problems.



Piecewise

$$\frac{1}{2}\int_0^b g^2(\theta)\,d\theta + \frac{1}{2}\int_b^{\frac{\pi}{2}} f^2(\theta)\,d\theta$$



$$R^2 - r^2$$

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p^2(\theta) - h^2(\theta)\,d\theta$$

Parametric Functions

The formula for finding the slope of a parametrized curve is:

$$\boxed{\frac{dy}{dx} = \frac{\;\frac{dy}{dt}\;}{\;\frac{dx}{dt}\;}}$$

This makes sense if we think about canceling dt.

To find the <u>second</u> derivative of a parametrized curve, we find the derivative of the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

1. Find the first derivative (*dy/dx*).

2. Find the derivative of *dy/dx* with respect to *t*.

3. Divide by *dx/dt*.

→

Find $\frac{d^2y}{dx^2}$ as a function of t if $x = t - t^2$ and $y = t - t^3$.

2. Find the derivative of dy/dx with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1-3t^2}{1-2t}\right) = \frac{2-6t+6t^2}{(1-2t)^2}$$

Quotient Rule

3. Divide by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{\mathrm{dx}}{\mathrm{dt}}} = \frac{\frac{2-6t+6t^2}{(1-2t)^2}}{1-2t}$$

$$= \frac{2-6t+6t^2}{(1-2t)^3}$$

Position, velocity, and acceleration in 2-dimensional space

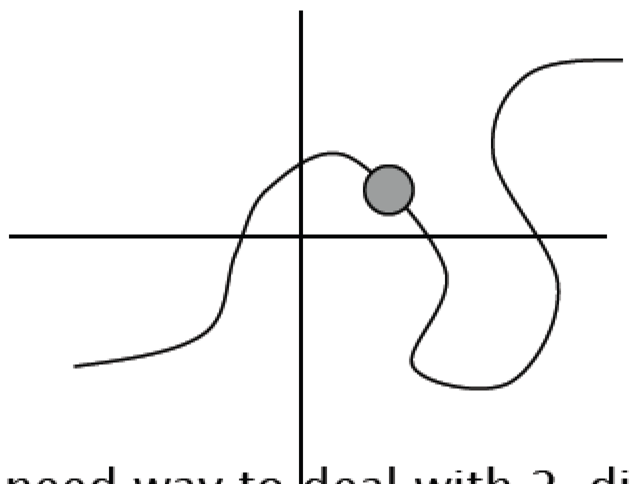
r(t) = Position at time *t*

v(t) = Velocity at time *t*

a(t) = Acceleration at time *t*

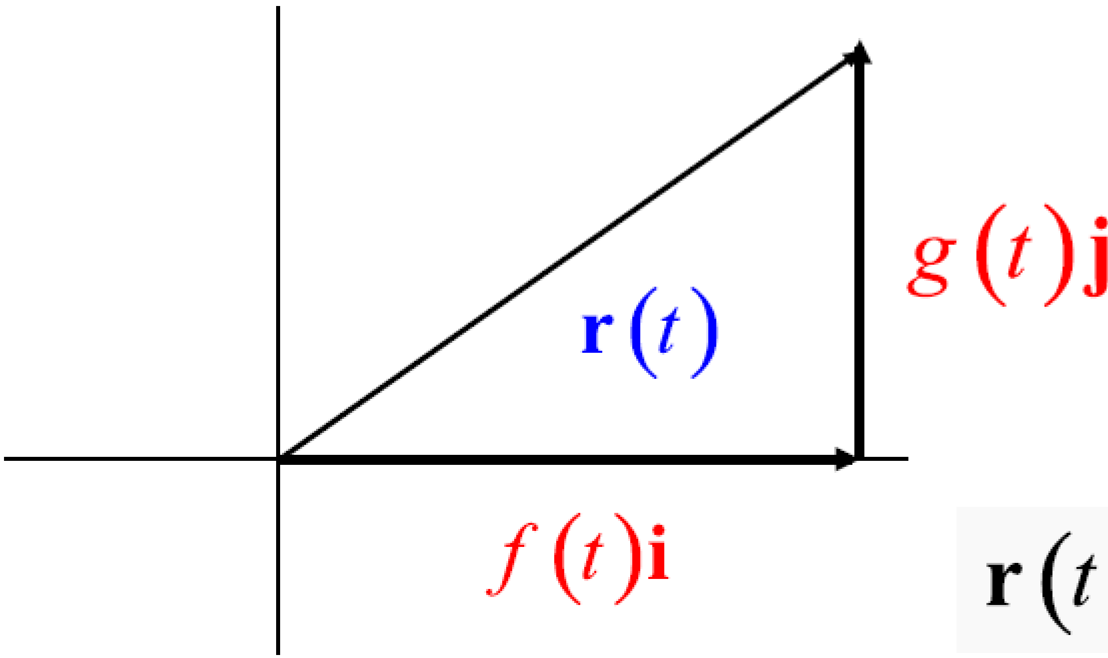
$$v(t) = \frac{dr}{dt}$$

$$a(t) = \frac{dv}{dt} = \frac{d^2 r}{dt^2}$$



Same idea, but we need way to deal with 2-dimensions

We can describe the position of a moving particle by a vector, $r(t)$.



$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$$

If we separate $r(t)$ into horizontal and vertical components, we can express $r(t)$ as a linear combination of standard unit vectors i and j.

Most of the rules for the calculus of vectors are the same as we have used, except:

$$\text{Speed} = \left|\mathbf{v}(\mathbf{t})\right|$$

"Speed" is magnitude of velocity. Speed has no direction.

$$\text{Direction} = \frac{\text{velocity vector}}{\text{speed}} = \frac{\mathbf{v}(t)}{\left|\mathbf{v}(t)\right|}$$

"Direction" is a unit vector that indicates direction but not magnitude.

→

Example 5:

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$$

a) Find the velocity and acceleration vectors.

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$$

b) Find the velocity, acceleration, speed and direction of motion at $t = \pi/4$

Example 5:

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$$

b) Find the velocity, acceleration, speed and direction of motion at $t = \pi/4$

velocity: $\mathbf{v}\left(\frac{\pi}{4}\right) = \left(-3\sin\frac{\pi}{4}\right)\mathbf{i} + \left(3\cos\frac{\pi}{4}\right)\mathbf{j} = -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j}$

acceleration: $\mathbf{a}\left(\frac{\pi}{4}\right) = \left(-3\cos\frac{\pi}{4}\right)\mathbf{i} - \left(3\sin\frac{\pi}{4}\right)\mathbf{j} = -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$

Example 5:

$$\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = (-3\sin t)\mathbf{i} + (3\cos t)\mathbf{j} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt} = (-3\cos t)\mathbf{i} - (3\sin t)\mathbf{j}$$

b) Find the velocity, acceleration, speed and direction of motion at $t = \pi / 4$

$$\mathbf{v}\left(\frac{\pi}{4}\right) == -\frac{3\sqrt{2}}{2}\mathbf{i} + \frac{3\sqrt{2}}{2}\mathbf{j} \qquad \mathbf{a}\left(\frac{\pi}{4}\right) == -\frac{3\sqrt{2}}{2}\mathbf{i} - \frac{3\sqrt{2}}{2}\mathbf{j}$$

speed: $$\left|\mathbf{v}\left(\frac{\pi}{4}\right)\right| = \sqrt{\left(-\frac{3\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2} = \sqrt{\frac{18}{4} + \frac{18}{4}} = 3$$

direction: $$\boxed{\frac{\mathbf{v}(\pi/4)}{|\mathbf{v}(\pi/4)|}} = \frac{-3\sqrt{2}/2}{3}\mathbf{i} + \frac{3\sqrt{2}/2}{3}\mathbf{j} = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

Curve lengths

Rectangular

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Polar

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Parametric

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$