Day 3 Integration Review

MULTIPLE-CHOICE QUESTIONS

A calculator may not be used for the following questions.

1. $\int \sec^2 x - 2\,dx =$

B

(A) ~~$\tan x + C$~~
(B) $\boxed{\tan x - 2x + C}$
(C) $\frac{\tan^3 x}{3} - x + C$
(D) $\frac{\sec^3 x}{3} - 2x + C$

2. Find $\int_0^2 3x^2 f(x^3)\,dx$ if $\int_0^8 f(t)\,dt = k$.

D

(A) k^3
(B) $9k$
(C) $3k$
(D) k

$u = x^3$
$du = 3x^2\,dx$
$u(2) = 8$
$u(0) = 0$
$\int f(u)\,du$

3. If $F(x) = \int_0^{x^2} \sqrt{t+5}$, what is $F'(2)$

C

(A) $\sqrt{7}$
(B) 3
(C) 12
(D) 18

$F'(x) = \left(\sqrt{x^2+5}\right) 2x$
$F'(2) = \sqrt{9} \cdot 2(2)$

4. What is the average value of $f(x) = (\sin x)4^{\cos x}$ for the closed interval $0 \le x \le \frac{\pi}{2}$?

B

(A) $\frac{3}{\ln 4}$
(B) $\frac{6}{\pi \ln 4}$
(C) $\frac{4}{\ln 4}$
(D) $\frac{8}{\pi \ln 4}$

$f_{av} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} (\sin x)\,4^{\cos x}\,dx$

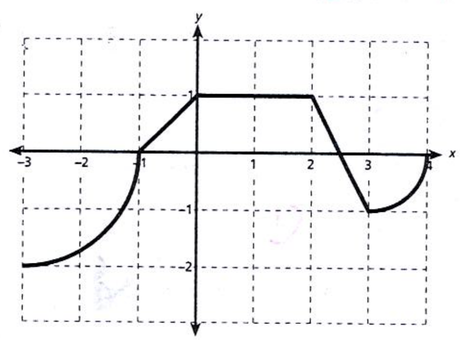
$\frac{2}{\pi} \int_1^0 -4^u\,du$

$\frac{2}{\pi}\left[\frac{-4^u}{\ln 4}\Big|_1^0\right] = \frac{2}{\pi}\left[\frac{-1}{\ln 4} + \frac{+4}{\ln 4}\right] = \frac{6}{\pi \ln 4}$

5. The graph of $f(x)$ consists of line segments and quarter circles as shown in the graph to the right. What is the value of $\int_{-3}^{4} f(x)\,dx$?

A

(A) $\frac{10 - 5\pi}{4}$
(B) $\frac{10 + 5\pi}{4}$
(C) $\frac{12 + 5\pi}{4}$
(D) $\frac{12 - 5\pi}{4}$



$-\frac{1}{4}\pi 2^2 + \frac{1}{2}(1)(1) + 2(1) + \frac{1}{2}\left(\frac{1}{2}\right)(1) - \frac{1}{2}\left(\frac{1}{2}\right)(1) - \frac{1}{4}\pi 1^2$

$-\pi + \frac{1}{2} + 2 + \cancel{\frac{1}{4} - \frac{1}{4}} - \frac{\pi}{4}$

$-\frac{5}{4}\pi + \frac{5}{2}$

A calculator may be used for the following questions.

6. Let R be the region between the function $f(x) = x^3 + 6x^2 + 10x + 4$, the x-axis, and the lines $x = 0$ and $x = 4$. Using the Trapezoidal Sum, compute the area when there are four equal subdivisions.

C

(A) 196
(B) 288
(C) 296
(D) 396

x	0	1	2	3	4
y	4	21	56	115	204

$\frac{1}{2}\left[4 + 2(21) + 2(56) + 2(115) + 204\right]$

7. What is $f(x)$ if $f'(x) = \frac{x}{x^2 - 1}$ and $f(2) = 0$?

B

(A) $f(x) = \frac{1}{2}\ln|x^2 - 1| - \ln 3$
(B) $f(x) = \frac{1}{2}\ln|x^2 - 1| - \ln\sqrt{3}$
(C) $f(x) = \frac{1}{2}\ln|x^2 - 1| + \ln\sqrt{3}$
(D) $f(x) = \frac{1}{2}\ln|x| - \frac{1}{2}x$

$\frac{1}{2}\int \frac{2x}{x^2 - 1}\,dx$

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|x^2 - 1| + C$

$0 = \frac{1}{2}\ln|3| + C$

$C = -\ln\sqrt{3}$

8. What is the average value of $f(x) = 2\ln x$? on the closed interval $1 \le x \le 3$

C

(A) 2.592
(B) 2.000
(C) 1.296
(D) 1.952

$\frac{1}{3-1}\int_1^3 2\ln x\,dx$

9. Evaluate: $\int_{-1}^{0} \frac{x^2}{\sqrt[5]{2x^3 + 1}}\,dx$.

D

(A) $\frac{4}{15}$
(B) $\frac{5}{12}$
(C) 0
(D) The function is not integrable on the interval $-1 \le x \le 0$.

$\frac{1}{6}\int_{-1}^{1} u^{-1/5}\,du = \frac{1}{6}\,u^{4/5} \cdot \frac{5}{4}\Big]_{-1}^{1}$

$\frac{5}{24}$ $-\frac{5}{24}$

Denom $= 0$
$x = \sqrt[3]{-\frac{1}{2}}$

10. If $\int_1^3 f(x)\,dx = k$ and $\int_1^7 f(x)\,dx = -4$, what is the value of $\int_7^3 x + f(x)\,dx$?

D

(A) $k - 4$
(B) $16 - k$
(C) ~~$-16 - k$~~
(D) $-16 + k$

$\int_7^3 x + \int_7^3 f(x)$

$-\frac{1}{2}x^2\Big]_3^7 + 4 + k$

$\frac{49}{-2} + \frac{9}{+2} + 4 + k$

$\int_1^3 + \int_3^7 = \int_1^7$

$k + \int_3^7 = -4$

$\int_3^7 = -4 - k$

x	−4	−3	−2	−1	0	1	2	3	4
f(x)	0.48	1.25	1.07	0.53	0.27	1.04	3.56	2.18	2

11. Selected values for the continuous function $f(x)$ are given in the table above. Using three left-hand rectangles of equal width, an approximation for $\int_{-3}^{3} f(x)\,dx$ is

C

(A) 9.90
(B) 7.72
(C) 5.64
(D) 4.90

$1.25(2) + .53(2) + 1.04(2)$

12. Initially a water tank contains 100 ft^3. Water begins to drain from the tank at the rate of $\frac{30e^{\frac{t}{3}}}{1+e^{\frac{t}{3}}}$ cubic feet per hour. How many cubic feet of water will remain in the tank after 3 hours?

(A) 39.768
(B) 44.190
(C) 55.810
(D) 60.232

B

$100 - \int_0^3 \frac{30e^{t/3}}{1+e^{t/3}}\,dt$

A calculator may not be used on the following questions.

13. If $R(x)$ is an even function and $S(x)$ is an odd function where $\int_{-a}^{0} R(x)dx = 5$ and $\int_0^a S(x)dx = -3$, find the value of $\int_{-a}^{a}[2(S(x)) - R(x) + 3]\,dx$.

(A) $-10+3a$
(B) $-10+6a$
(C) $10+6a$
(D) $-6+6a$

B

$2(0) - 2(5) + 3x\Big]_{-a}^{a}$

$-10 + 6a$

14. $\int \frac{9x}{\sqrt{1-81x^4}}\,dx =$

(A) $(1-9x^2)^{\frac{1}{2}} + C$
(B) $\frac{1}{2}(1-9x^2)^{\frac{3}{2}} + C$
(C) $\sin^{-1}(9x^2) + C$
(D) $\frac{1}{2}\sin^{-1}(9x^2) + C$

D

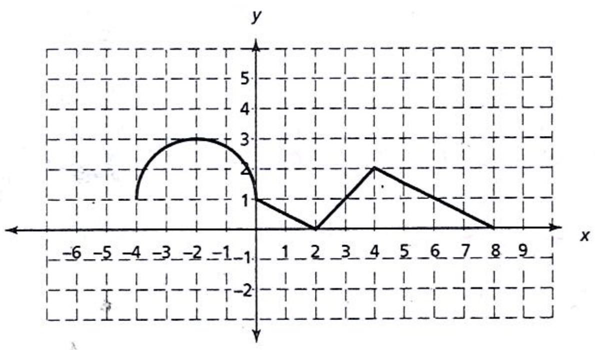
$\frac{1}{2}\int \frac{9x}{\sqrt{1-(9x^2)^2}}$

$u = 9x^2$

$du = 18x$

$\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}$

$\frac{1}{2}\sin^{-1}u + C$



15. The graph of $f(x)$ shown above consists of three line segments and one semicircle. Let $g(x) = \int_{-2}^{x} f(t)dt$. Which of the following statements must be false?

(A) $g(4) = \int_{-2}^{4} f(t)dt = \pi + 5$
(B) $g(-4) = \int_{-2}^{-4} f(t)dt = -\pi - 2$
(C) $g'(6) = 1$
(D) $g(x)$ has a relative maximum at $x = 4$.

D

$g(4) = \frac{1}{4}\pi 2^2 + 2 + \frac{1}{2}(2)(1) + \frac{1}{2}(2)(2)$

$g(-4) = -\int_{-4}^{-2} f(t)dt = -(\frac{1}{4}\pi 2^2 + 2)$

$g' = f \quad f(6) = 1$

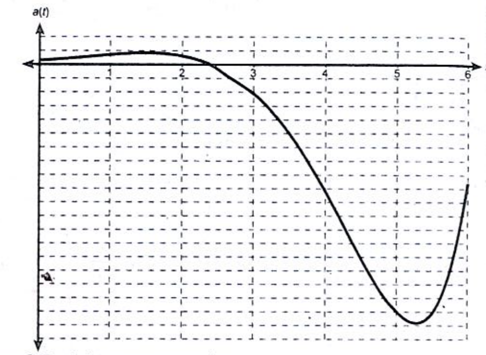
$g' = f \quad f(4) \neq 0$

$\therefore$ no extrema

FREE-RESPONSE QUESTION

This question requires the use of a calculator.

1. The acceleration of a particle is given as $a(t) = 3e^x - x^4$ cm/sec^2 on the closed interval [0, 3] as illustrated on the graph shown.
   (a) Find the velocity of the particle at any time t if $v(0) = 6$ cm/sec.
   (b) Find the position of the particle at any time t if $x(0) = -5$ cm.
   (c) At $t = 6.5$, is the speed of the particle increasing or decreasing? Explain your reasoning.
   (d) On the closed interval [0, 3], what is the velocity of the particle when its acceleration is at a maximum? Explain your reasoning.



a) $\int a(t)dt = v(t)$

$\int 3e^x - x^4 dx = 3e^x - \frac{1}{5}x^5 + C$

$6 = 3e^0 - \frac{1}{5}(0)^5 + C$

$6 = 3 + C$

$3 = C$

$v(t) = 3e^x - \frac{1}{5}x^5 + 3$

b) $\int v(t) = x(t)$

$\int 3e^t - \frac{1}{5}t^5 + 3\,dt = 3e^t - \frac{1}{5\cdot 6}t^6 + 3t + C$

$-5 = 3e^0 - 0 + 0 + C$

$-8 = C$

$x(t) = 3e^t - \frac{1}{30}t^6 + 3t - 8$

c) $a(6.5) = 210.362$

$v(6.5) = -322.156$

b/c acceleration is pos, but velocity is negative the speed is decreasing.

d) $a'(t) = 3e^t - 4t^3 = 0$

$t = 1.496$

$v(1.496) = 14.892$

$a(0) = 3$

$a(1.496) = 8.382$ max

$a(3) = -20.743$

MULTIPLE-CHOICE QUESTIONS

A calculator may not be used for the following questions.

1. $\int 9x\cos(3x+1)dx =$
   (A) $9x\cos(3x+1)+\sin(3x+1)+C$
   (B) $3x\sin(3x+1)-\cos(3x+1)+C$
   (C) $3\sin(3x+1)-\frac{1}{3}\cos(3x+1)+C$
   (D) $3x\sin(3x+1)+\cos(3x+1)+C$

$u = 9x \quad dv = \cos(3x+1)$
$du = 9 \quad v = \frac{1}{3}\sin(3x+1)$
$= 3x\sin(3x+1) - \int 3\sin(3x+1)$
$3x\sin(3x+1)+\cos(3x+1)+C$

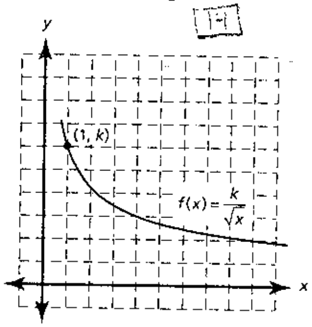
x	g(x)	g'(x)	g"(x)
0	2	−3	5
1	3	8	−2

3. Let g be a twice-differentiable function with the selected values of g and its derivatives shown in the table above. What is the value of $\int_0^1 2xg''(x)dx$?
   (A) 10
   (B) 12
   (C) 8
   (D) −4

$u = 2x \quad dv = g''(x)$
$du = 2 \quad v = g'(x)$
$2xg'(x) - \int 2g'(x)$
$2xg'(x) - 2g(x)\Big]_0^1 = 2(1)(8) - 2(3) - [\ldots]$
$16 - 6 + 4$
$\boxed{14}$

4. Let $f(x) = \frac{k}{\sqrt{x}}$, where $x > 0$ and k is some finite positive constant, as pictured at the right.
   Let $L = \int_0^1 f(x)dx$ and $M = \int_1^\infty f(x)dx$. Which one of the following statements is true?
   (A) $L < M$
   (B) $L = M$
   (C) $L > M$
   (D) No conclusion can be made about the relative values of L and M.

A



$L = \lim_{a\to 0} \int_a^1 \frac{k}{\sqrt{x}} = \lim_{a\to 0} 2kx^{1/2}\Big]_a^1 = \lim_{a\to 0} 2k - 2k\sqrt{a}$
$M = \lim_{b\to\infty} \int_1^b \frac{k}{\sqrt{x}} = \lim_{b\to\infty} 2kx^{1/2}\Big]_1^b = \lim_{b\to\infty} 2k\sqrt{b} - 2k$
$L = 2k \quad M = \infty$

6. $\int \frac{4dx}{(x-3)(x+1)} =$
   (A) $\ln|x-3|+\ln|x+1|+C$
   (B) $\ln\left|\frac{x-3}{x+1}\right|+C$
   (C) $4\ln|(x-3)(x+1)|+C$
   (D) $4\ln\left|\frac{x-3}{x+1}\right|+C$

B

$\frac{4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$
$4 = A(x+1) + B(x-3)$
$x = -1 \quad x = 3$
$4 = B(-4) \quad 4 = A(4)$
$-1 = B \quad 1 = A$
$= \ln|x-3| - \ln|x+1| + C$

8. Let $f(x)$ be a differentiable function with the properties that $f(1) = 5$ and $\lim_{x\to\infty} f(x) = -8$. $\int_1^\infty f'(x)dx =$
   (A) −13
   (B) −8
   (C) 5
   (D) ∞

A

$\lim_{a\to\infty} \int_1^a f'(x)\,dx$
$\lim f(x)\Big]_1^a$
$\lim_{x\to\infty} f(a) - f(1)$
$-8 - 5$
-13

$\frac{2}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{(x+1)}$

10. $\int \frac{2}{x^2+4x+3}dx = \int \frac{2}{(x+3)(x+1)}$
   (A) $\ln|x+1|+\ln|x+3|+C$
   (B) $(\ln|x+1|)(\ln|x+3|)+C$
   (C) $\ln\left|\frac{x+1}{x+3}\right|+C$
   (D) $\frac{\ln|x+1|}{\ln|x+3|}+C$

C

$2 = A(x+1) + B(x+3)$
$x = -1 \quad x = -3$
$2 = B(2) \quad 2 = A(-2)$
$1 = B \quad -1 = A$
$\int = -\ln|x+3| + \ln|x+1|$

12. $\int 2x^2e^{3x}dx =$
   (A) $\frac{2}{3}x^2e^{3x} - \frac{4}{9}xe^{3x} + \frac{4}{27}e^{3x} + C$
   (B) $2x^2e^{3x} + 4xe^{3x} + C$
   (C) $6x^2e^{3x} - 36xe^{3x} + 108e^{3x} + C$
   (D) $\frac{2}{3}x^2e^{3x} - \frac{4}{3}xe^{3x} + \frac{4}{3}e^{3x} + C$

A

$\frac{2}{3}x^2e^{3x} - \frac{4}{9}xe^{3x} + \frac{4}{27}e^{3x} + C$

13. $f(x)$ is a twice differentiable function on $x \in [2,6]$ with selected values given in the table below.

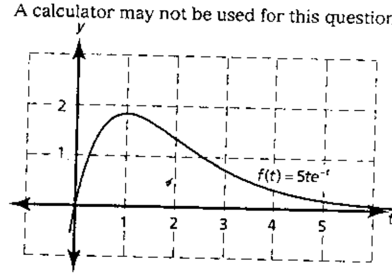
x	f(x)	f'(x)	f"(x)
2	8	1	−1
6	3	2	3

$\int_2^6 x \cdot f''(x)dx =$
   (A) 15
   (B) 20
   (C) 26
   (D) 34

$xf'(x) - f(x)\Big]_2^6$
$6f'(6) - f(6) - [2f'(2) - f(2)]$
$6\cdot 2 - 3 - [2 - 8]$
$12 - 3 - 2 + 8$
15

FREE-RESPONSE QUESTION

A calculator may not be used for this question.



Let $F(x) = \int_0^x 5te^{-t}dt$ for $t \ge 0$ and $x \ge 0$.
(a) Find an expression for $F'(x)$ in terms of x only.
(b) Is the graph of $F(x)$ concave up or concave down at $x = 2$? Explain your answer.
(c) Find an expression for $F(x)$, in terms of x only, that does not involve an integral.
(d) Using your answer to c, find $\lim_{x\to\infty} F(x)$. Justify your answer.
(e) Using your answer to d, explain what is meant by the expression $\lim_{x\to\infty} F(x)$.

a) $5xe^{-x} = F'(x)$
b) $F''(x) = 5e^{-x} + 5xe^{-x}(-1)$
$F''(2) = \frac{5}{e^2} - \frac{10}{e^2} = \frac{-5}{e^2}$
neg, ∴ CD

c) $u = 5t \quad dv = e^{-t}$
$du = 5 \quad v = -e^{-t}$
$F(x) = -5te^{-t} + \int 5e^{-t}$
$F(x) = -5te^{-t} - 5e^{-t}\Big]_0^x$
$F(x) = -5xe^{-x} - 5e^{-x} - (0 - 5)$

d) $\lim_{x\to\infty} \frac{-5x}{e^x} - \frac{5}{e^x} + 5$
$\overset{H}{=} \lim_{x\to\infty} \frac{-5}{e^x} - 0 + 5$
$= 0 - 0 + 5 = 5$

left and right of 4, there is no relative maximum at $x = 4$.
(*Calculus for AP* 1st ed. pages 317–325; EU 3.2,3.3; LO 3.2C,3.3A; EK 3.2C1,3.3A2; MPAC 2,3,4)

FREE-RESPONSE QUESTION

	Solution	Possible points
(a)	Given: $a(t) = 3e^t - t^4$; $v(0) = 6$ cm/s $v(t) = \int 3e^t - t^4 dx = 3e^t - \frac{t^5}{5} + C$ $v(0) = 6$ which equals $3 - 0 + C$; therefore, $C = 3$. Then, $v(t) = 3e^t - \frac{t^5}{5} + 3$ cm/s.	$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(b)	Since $x(t) = \int 3e^x - \frac{x^5}{5} + 3dx = 3e^x - \frac{x^6}{30} + 3x + C$ and $x(0) = -5 = 3 - 0 + 0 + C$; therefore $C = 8$. Then $x(t) = 3e^x - \frac{x^6}{30} + 3x - 8$ cm.	$2: \begin{cases} 1: \text{integrand} \\ 1: \text{answer} \end{cases}$
(c)	The speed of the particle is increasing when $a(t)$ and $v(t)$ have the same signs, and decreasing when $a(t)$ and $v(t)$ have opposite signs. Since $a(6.5) =$ 210.362 cm/sec^2 and $v(6.5) = -322.156$ cm/sec, the speed of the particle at $t = 6.5$ sec is decreasing.	$3: \begin{cases} 1: a(6.5) \\ 1: v(6.5) \\ 1: \text{reasoning} \end{cases}$
(d)	$a'(t) = 3e^x - 4x^3 = 0$ at $t = 1.496$ or $t = 5.279$ but $v(1.496) = 14.892$ cm/sec. only 1.496 is in our interval [0, 3] For $0 < t < 1.496$, $a'(t) > 0$, and for $1.496 < t < 3$, $a'(t) < 0$. Therefore, $a(t)$ changes from increasing to decreasing at $t = 1.496$, so a maximum acceleration occurs at $t = 1.496$ and its velocity at t = 1.496 is 14.892 cm/s	$2: \begin{cases} 1: \text{answer} \\ 1: \text{reasoning} \end{cases}$

(a), (b) (*Calculus for AP* 1st ed. pages 317–325, 345–346; EU 3.4,3.3; LO 3.4A,3.3B; EK 3.4A2,3.3B2; MPAC 2,3,4)
(c), (d) (*Calculus for AP* 1st ed. pages 147–153; EU 3.4; LO 3.4C; EK 3.4C1; MPAC 2,3)

15. ANSWER: (A) Using cross sections and recognizing the improper integral,

$$V = \lim_{k\to 0^+}\int_k^{16}\left(x^{-\frac{1}{4}}\right)^2 dx = \lim_{k\to 0^+}\int_k^{16} x^{-\frac{1}{2}}dx = \lim_{k\to 0^+} 2x^{\frac{1}{2}}\Big|_k^{16} = \lim_{k\to 0^+}(8-k) = 8.$$

(*Calculus for AP* 1st ed. pages 423–424, 515–521; EU 3.2,3.4; LO 3.2D,3.4D; EK 3.2D1,3.2D2,3.4D2; MPAC 2,3)

FREE-RESPONSE QUESTION

	Solution	**Possible points**
(a)	By the Second Fundamental Theorem, $F'(x) = 5xe^{-x}$.	1: answer
(b)	Using the Product Rule, $F''(x) = \frac{d}{dx}[F'(x)] = 5x\cdot -e^{-x} + 5e^{-x} = 5e^{-x}(1-x)$ $F''(2) = 5e^{-2}(1-2) < 0$; therefore the graph of $F(x)$ is concave down at $x = 2$.	2: $\begin{cases}1\text{: } F''(x)\\ 1\text{: answer with reason}\end{cases}$
(c)	Integration by parts $u = 5t$ $\quad v = -e^{-t}$ $du = 5\,dt$ $\quad dv = e^{-t}\,dt$ $F(x) = \int_0^x 5te^{-t}\,dt = -5te^{-t}\Big\|_0^x + \int_0^x 5e^{-t}\,dt$ $= -5te^{-t} - 5e^{-t}\Big\|_0^x = \left(-5xe^{-x} - 5e^{-x}\right) - (0-5)$ $= -5e^{-x}(x+1) + 5$	3: $\begin{cases}1\text{: parts setup}\\ 1\text{: antiderivative}\\ 1\text{: answer}\end{cases}$
(d)	$\lim_{x\to\infty}\left[-5e^{-x}(x+1)+5\right] = 5 + \lim_{x\to\infty}\frac{-5(x+1)}{e^x}$ Using L'Hôpital's Rule, this is equal to $5 + \lim_{x\to\infty}\frac{-5}{e^x} = 5 + 0 = 5$.	2: $\begin{cases}1\text{: recognition of indeterminate form}\\ 1\text{: answer}\end{cases}$
(e)	Since $F(x) = \int_0^x 5te^{-t}\,dt$ for $t \ge 0$ and $x \ge 0$. If $x \Rightarrow \infty$, then this becomes the improper integral $\lim_{x\to\infty}\int_0^x 5te^{-t}dt$, which will yield the area from $x = 0$ to $x = \infty$ between the graph of $f(t)$ and the horizontal axis. Based on the limit value in part c, this area is 5.	1: answer

(a) (*Calculus for AP* 1st ed. pages 323–325; EU 3.3; LO 3.3A; EK 3.3A2; MPAC 1,2)
(b) (*Calculus for AP* 1st ed. pages 147–148; EU 2.2; LO 2.2A; EK 2.2A1; MPAC 2,3)
(c) (*Calculus for AP* 1st ed. pages 461–466; EU 3.3; LO 3.3B; EK 3.3B5; MPAC 2,3)
(d) (*Calculus for AP* 1st ed. pages 504–510; EU 1.1; LO 1.1C; EK 1.1C3; MPAC 2,3)
(e) (*Calculus for AP* 1st ed. pages 515–521; EU 3.2; LO 3.2D; EK 3.2D1,3.2D2; MPAC 2,3)