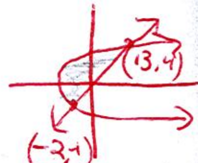
 Day 5 Applications of Integration

MULTIPLE-CHOICE QUESTIONS

A calculator may not be used for the following questions.

For Questions 1 and 2, region R is bounded by $f(y) = y^2 - 3$ and $g(y) = 3y + 1$.



right − left

C

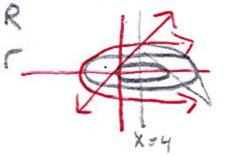
1. Which of the following expressions gives the area of region R?
   ~~(A)~~ $\int_{-2}^{13} (3y+1) - (y^2 - 3)\, dy$
   ~~(B)~~ $\int_{-2}^{13} (y^2 - 3) - (3y+1)\, dy$
   (C) $\int_{-1}^{4} (3y+1) - (y^2 - 3)\, dy$
   (D) $\int_{-1}^{4} (y^2 - 3) - (3y+1)\, dy$

C

2. Which of the following expressions gives the volume when region R is rotated about the line $x = 4$?
   (A) $\pi \int_{-2}^{13} (7 - y^2)^2 - (3 - 3y)^2\, dy$
   ~~(B)~~ $\pi \int_{-1}^{4} (1 - y^2)^2 - (5 - 3y)^2\, dy$
   (C) $\pi \int_{-1}^{4} (7 - y^2)^2 - (3 - 3y)^2\, dy$
   (D) $\pi \int_{-2}^{13} (3y+1)^2 - (y^2 - 3)^2\, dy$

$4 - (y^2 - 3) = R$

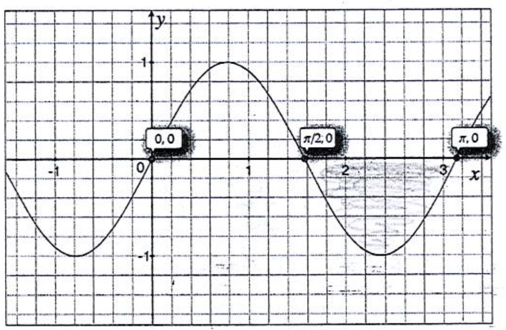
$4 - (3y+1) = r$



For problems 3 and 4, region Q is bounded by $y = \sin 2x$, $y = 0$, $x = \frac{\pi}{2}$ and $x = \pi$.

D

3. What is the area of region?
   (A) 0
   (B) $\frac{1}{2}$
   (C) $\frac{\pi}{2}$
   (D) 1



$\int_{\frac{\pi}{2}}^{\pi} 0 - \sin 2x\, dx$

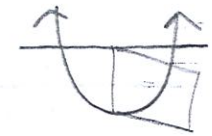
$+\frac{1}{2}\cos 2x\Big]_{\pi/2}^{\pi} = \frac{1}{2}\cos 2\pi - \frac{1}{2}\cos \pi$

$\frac{1}{2}(1) - \frac{1}{2}(-1)$

$\frac{1}{2} + \frac{1}{2}$

B

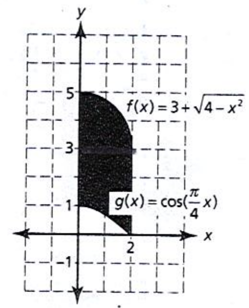
4. Which of the following expressions gives the volume of a solid whose base in the xy-plane is region Q and whose cross sections, perpendicular to the x-axis, are squares with a side in the xy-plane?
   (A) $\pi \int_{\frac{\pi}{2}}^{\pi} (1 - \cos^2 2x)\, dx$
   (B) $\int_{\frac{\pi}{2}}^{\pi} (-\sin 2x)^2\, dx$
   (C) $\pi \int_{\frac{\pi}{2}}^{\pi} (1 - \cos 2x)\, dx$
   (D) $\pi \int_{\frac{\pi}{2}}^{\pi} (-\sin 2x)^2\, dx$



Area $= s^2$

Side $= -\sin 2x$

For Question 5, region W is bounded by $f(x) = 3 + \sqrt{4 - x^2}$, $g(x) = \cos\left(\frac{\pi}{4}x\right)$, $x = 0$, and $x = 2$.



$\int_0^2 3 + \sqrt{4 - x^2} - \cos\left(\frac{\pi}{4}x\right) dx$

¼ circle + rectangle

$\frac{1}{4}\pi(2)^2 + 2(3)$

$\pi + 6 \quad - \int_0^2 \cos\left(\frac{\pi}{4}x\right) dx$

$-\frac{4}{\pi}\sin\left(\frac{\pi}{4}x\right)\Big]_0^2$

$-\left(\frac{4}{\pi}\sin\frac{\pi}{2} - \frac{4}{\pi}\sin 0\right)$

$-\frac{4}{\pi} + 0$

B

5. What is the area of the region W?
   (A) 6.000
   (B) $6 + \pi - \frac{4}{\pi}$
   (C) $6 + \pi + \frac{4}{\pi}$
   (D) $6 + 2\pi - \frac{4}{\pi}$

$\int_0^2 3+\sqrt{4-x^2}-3 = \int_0^2 k\cos\frac{\pi}{4}$

6. In the closed interval $0 \le x \le 2$, if $h(x) = k\cos\left(\frac{\pi}{4}x\right)$, for what value of k does the region bounded by $f(x)$ and the line $y = 3$ have the same area as that bounded by $h(x)$ and the x-axis?

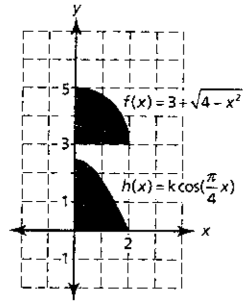
C

(A) $\frac{1}{2}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi^2}{4}$

(D) $\frac{\pi^2}{2}$



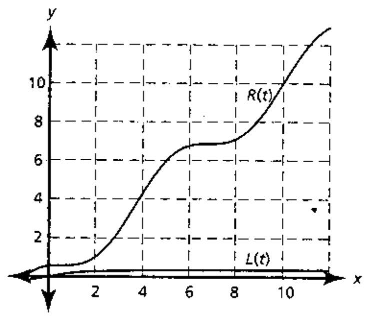
Use #5

A calculator may be used on the following questions.

$\pi = k\int_0^2 \cos\frac{\pi}{4}$

$\pi = k\left(\frac{4}{\pi}\right)$

$\frac{\pi^2}{4} = k$



7. Air is being pumped into a spherical balloon at the rate of $R(t) = t + \cos(t+1)\,\text{cm}^3/\text{min}$, but there is a small hole in the balloon and at the same time air is leaking out at the rate of $L(t) = 0.25\tan^{-1}(t)\,\text{cm}^3/\text{min}$. Which of the following is true for $0 \le t \le 10$, where t is measured in minutes?

C

(A) $\int_0^{10} R(t) - L(t)dt$ represents the volume of the balloon after the first 10 minutes.

(B) $\frac{1}{10}\int_0^{10} R(t) - L(t)dt$ represents the increase in volume each minute during the first 10 minutes.

(C) $\int_0^{10} R(t) - L(t)dt$ represents the increase in volume during the first 10 minutes.

(D) $R(0) - L(0) = 1$

Since initial volume is not known

8. The revenue and expenditures for a small company are analyzed and predicted for the next 5 years. The current annual revenue is \$125,000 and growing at an annual rate of 30%, while the expenditures are modeled with a sinusoidal function. If

$R(t) = 125{,}000(1.3)^t$ and $E(t) = 25{,}000e^{\frac{t}{2}}\left(\sin\left(\frac{t}{2}\right)+\cos\left(\frac{t}{3}\right)\right)$,

max of $E(t)$ at $t = 4.077$

D

to the nearest dollar, what is the average annual profit for the company when the expenditures reach a maximum value on the interval $0 \le t \le 5$?

$\frac{1}{4.077}\int_0^{4.077} R(t) - E(t) =$

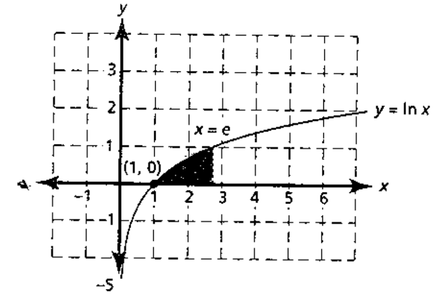
(A) \$434,691
(B) \$340,827
(C) \$127,282
(D) \$106,613

9. Region G is bounded by the curve $y = \ln x$, $x = e$, and the x-axis. Order from smallest to largest the volumes determined when G is rotated about the axes:

I. $y = 0$ II. $y = 1$ III. $y = e$

C

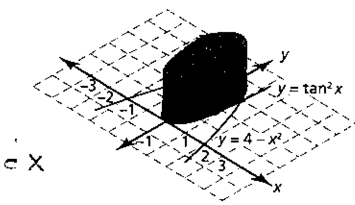
(A) III < I < II
(B) II < I < III
(C) I < II < III
(D) I < III < II



10. The base of a solid is bounded by $y = \tan^2 x$ and $y = 4 - x^2$ in the xy-plane, as shown in the figure below. Each cross section perpendicular to the x-axis is a rectangle with one side in the x-y plane and whose height is 2. What is the volume of the solid?

D

(A) 3.121
(B) 4.454
(C) 6.243
(D) 12.486



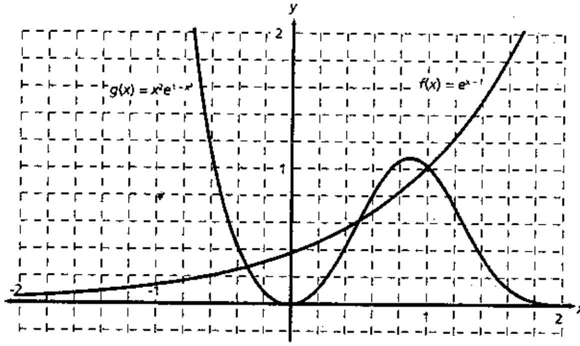
$\int_{-1.041}^{1.041} 2(4-x^2-\tan^2 x)\,dx$

9) I. $\pi\int_1^e (\ln x)^2 dx \approx 2.257$

II $\pi\int_1^e 1^2-(1-\ln x)^2 dx \approx 4.027$

III $\pi\int_1^e e^2-(e-\ln x)^2 dx \approx 14.2$

A calculator is required for the following questions.



11. Find the combined area of the two regions bounded by the curves $f(x) = e^{x-1}$ and $g(x) = x^2e^{1-x^3}$ as shown in the figure above.
   (A) 0.073
   (B) 0.203
   (C) 0.276
   (D) 0.931

C

$\int_{-.3103}^{.5051} e^{x-1} - x^2 e^{1-x^3} + \int_{.5051}^{1} x^2 e^{1-x^3} - e^{x-1}$

Questions 12–14 refer to the following information:

The velocity of a particle moving along a line is given by $v(t) = e^t \sin(e^t)$ for the interval $0 \le t \le \frac{\pi}{2}$.

12. How far to the right of the starting point will the particle be at $t = \frac{\pi}{2}$?
   (A) 0.442
   (B) 0.540
   (C) 1.145
   (D) 1.540

$\int_0^{\pi/2} e^t \sin(e^t)$

13. What is the distance that the particle has traveled on this time interval?
   (A) 0.442
   (B) 1.145
   (C) 1.540
   (D) 2.638

$\int_0^{\pi/2} |e^t \sin(e^t)|$

14. If the position of the particle at time $t = 0$ is -1, then what is the position of the particle when it is farthest to the right on this time interval?
   (A) 0.442
   (B) 0.540
   (C) 1.145
   (D) 1.540

B

$v(t) = 0$

$t = 1.1447$ $\int_0^{1.1447} v(t) = p(1.1447) - p(0)$

$1.540 = p(1.1447) - (-1)$

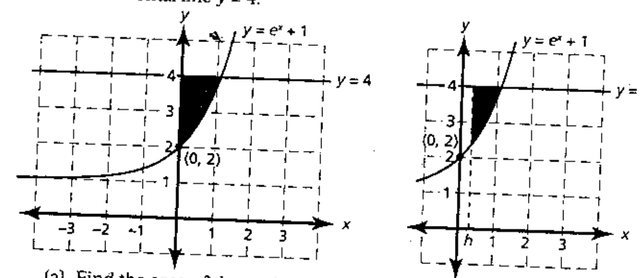
15. Which of the following expressions represents the volume of the solid generated when the region bounded by $y = x^2 - 2x + 3$ and $y = 1 + 3x - x^2$ is rotated about the line $y = -2$?
   (A) $V = \pi\int_{\frac{1}{2}}^{2}\left[(1+3x-x^2)-(x^2-2x+3)\right]dx$
   (B) $V = \pi\int_{\frac{1}{2}}^{2}\left[(-1+3x-x^2)^2-(x^2-2x+1)^2\right]dx$
   (C) $V = \pi\int_{\frac{1}{2}}^{2}\left[(3+3x-x^2)^2-(x^2-2x+5)^2\right]dx$
   (D) $V = \pi\int_{\frac{1}{2}}^{2}\left[(1+3x-x^2)-(x^2-2x+3)\right]^2 dx$

C

FREE-RESPONSE QUESTION

A calculator may be used for this free-response question.

Let f be the function given by $f(x) = e^x + 1$ as shown in the sketch below, where the region R is bounded by the graph of $f(x)$, the y-axis, and the horizontal line $y = 4$.



(a) Find the area of the region R.
(b) A vertical line $x = h$, where $h > 0$ is chosen so that the area of the region bounded by $f(x)$, the y-axis, the horizontal line $y = 4$, and the line $x = h$ is half the area of region R. What is the value of h?
(c) Find the volume of the solid formed when region R is rotated about the line $y = 4$.
(d) A horizontal line $y = k$, where k is greater than 4, is chosen so that the volume of the solid formed when region R is rotated about the line $y = k$ is twice the volume of the solid found in part (c). Set up, but do *not* evaluate, an integral expression in terms of a single independent variable which represents the volume of this solid.

No Calculator

B

2. Which of the following integral expressions represents the area of the region bounded by the graphs of $y = \frac{9}{x^2+x-20}$, $x=-3$, $x=2$, and the x-axis?

(A) $\left|\int_{-3}^{2}\left(\frac{1}{x-4}+\frac{1}{x+5}\right)dx\right|$

(B) $\left|\int_{-3}^{2}\left(\frac{1}{x-4}-\frac{1}{x+5}\right)dx\right|$

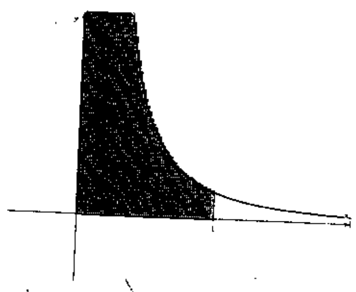
(C) $\left|\int_{-3}^{2}\left(\frac{5}{x-4}+\frac{4}{x+5}\right)dx\right|$

(D) $\left|\int_{-3}^{2}\left(\frac{1}{x+4}-\frac{1}{x-5}\right)dx\right|$

D

5. What is the area of the region in the first quadrant between the x-axis, y-axis, $y=\frac{1}{x^2}$, and $x=1$.

(A) 1
(B) 2
(C) 0
(D) ∞



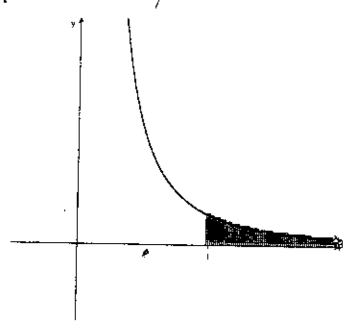
$\lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}dx$

$\lim_{a\to 0^+}\left(-\frac{1}{x}\Big|_a^1\right) = \lim_{a\to 0^+}\left(-\frac{1}{1}-\frac{-1}{a}\right) = -1+\infty$

A

7. What is the area of the region in the first quadrant between the x-axis, the graph $y=\frac{1}{x^2}$, and $x=1$?

(A) 1
(B) 2
(C) 0
(D) ∞



$\lim_{a\to\infty}\int_1^a \frac{1}{x^2}dx$

$\lim_{a\to\infty}\left(-\frac{1}{x}\Big|_1^a\right) = \lim_{a\to\infty}\left(-\frac{1}{a}-\frac{-1}{1}\right) = 0+1$

A

9. The area of the region bounded by the graphs of $y = xe^x$, $x=0$, $x=\ln 2$, and the x-axis is

(A) $2\ln(2)-1$
(B) $2\ln(2)$
(C) $2\ln(2)+1$
(D) $[\ln(2)]^2-1$

$\int_0^{\ln 2} xe^x\,dx = xe^x - \int e^x\,dx$

A

11. The area of the Quadrant I region under the function $f(x)=\frac{e^x}{1+e^{2x}}$ is

(A) $\frac{\pi}{4}$
(B) 1
(C) $\frac{\pi}{2}$
(D) divergent

$\int_0^\infty \frac{e^x}{1+e^{2x}}dx$ $\quad u=e^x$

$\int_1^\infty \frac{1}{1+u^2}du = \tan^{-1}u\Big]_1^\infty = \frac{\pi}{2}-\frac{\pi}{4}$

D

14. Let R be the region bounded by the graphs of the function $f(x)=\frac{1}{\sqrt{x^2-1}}$, the x-axis, and the lines $x=p$ and $x=q$, where $1<p<q$. An expression for the volume of the solid generated by revolving R about the x-axis is

(A) $\int_p^q \frac{1}{x^2-1}dx$

(B) $\frac{\pi}{2}\int_p^q\left[\frac{1}{x-1}+\frac{1}{x+1}\right]dx$

(C) $\pi\int_p^q\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx$

(D) $\frac{\pi}{2}\int_p^q\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx$

$\pi\int_p^q [f(x)]^2 = \pi\int_p^q \frac{1}{x^2-1} = \pi\int_p^q \frac{1/2}{x-1}-\frac{1/2}{x+1}$

A

15. The base of a certain solid is the region bounded by the coordinate axes, the line $x=16$, and the graph of $y=x^{-\frac{1}{4}}$. For this solid, each cross section perpendicular to the x-axis is a square whose side is across the base. The volume of this solid is

(A) 8
(B) $\frac{32}{3}$
(C) 8π
(D) divergent

$\lim_{a\to 0^+}\int_a^{16}\left(x^{-1/4}\right)^2$

$\lim_{a\to 0^+}\int_a^{16} x^{-1/2}$

$\lim_{a\to 0^+} 2x^{1/2}\Big]_a^{16}$

$2(4)-2(0)$

14. ANSWER: (B) The particle moves to the right when $v(t) > 0$, which is on the interval $0 < t < 1.1447$. Thus
$s(1.1447) \approx -1 + \int_0^{1.1447} e^t \sin(e^t)\,dt = -1 + 1.540 = 0.540$.
(*Calculus for AP* 1st ed. pages 317–325; EU 3.4; LO 3.4E; EK 3.4E1; MPAC 2,3)

15. ANSWER: (C) The x-coordinates of the points of intersection are $x = \frac{1}{2}$ and 2. The radius for rotation is $r = 2 + y$. Therefore
$r_o = 1 + 3x - x^2 + 2 = 3 + 3x - x^2$ and $r_i = x^2 - 2x + 3 + 2 = x^2 - 2x + 5$.
The volume is found by the washer method where
$V = \pi \int_a^b (r_o^2 - r_i^2)\,dx$. Thus the volume is given by
$V = \pi \int_{\frac{1}{2}}^{2} \left[(3 + 3x - x^2)^2 - (x^2 - 2x + 5)^2\right] dx$. (*Calculus for AP* 1st ed. pages 418–424; EU 3.4; LO 3.4D; EK 3.4D2; MPAC 2,3)

FREE-RESPONSE QUESTION

	Solution	Possible points
(a)	**Point of intersection:** $e^x + 1 = 4 \Rightarrow x = \ln 3 \approx 1.099$ $A = \int_0^{1.099} \left[4 - (e^x + 1)\right] dx \approx 1.296$	1: Limits and constant in all parts $2: \begin{cases} 1: & \text{integrand} \\ 1: & \text{answer} \end{cases}$
(b)	$\int_0^h 4 - (e^x + 1)\ dx \approx 0.648$ $\left(3x - e^x\right)\Big\|_0^h = 3h - e^h - (0 - 1)$ $= 3h - e^h + 1 \approx 0.648$ $h \approx 0.361$	$2: \begin{cases} 1: & \text{antiderivative} \\ 1: & \text{answer} \end{cases}$
(c)	$V = \pi \int_0^{1.099} \left[4 - \left(e^x + 1\right)\right]^2 dx \approx 1.888\pi$, or 5.930	$2: \begin{cases} 1: & \text{integrand} \\ 1: & \text{answer} \end{cases}$
(d)	$V = \pi \int_0^{1.099} \left\{\left[k - \left(e^x + 1\right)\right]^2 - \left(k - 4\right)^2\right\} dx = 11.86$	$2: \begin{cases} 0/2 & \text{if not difference of two squares} \\ 1/2 & f^2 - g^2 \\ 1/2 & \text{if reversal} \end{cases}$

(a) (*Calculus for AP* 1st ed. pages 140, 345–346, 408–413; EU 3.4; LO 3.4D; EK 3.4D1; MPAC 2,3,4)
(b) (*Calculus for AP* 1st ed. pages 408–413; EU 3.4; LO 3.4D; EK 3.4D1; MPAC 2,3,4)
(c) (*Calculus for AP* 1st ed. pages 418–424; EU 3.4; LO 3.4D; EK 3.4D2; MPAC 2,3,4)
(d) (*Calculus for AP* 1st ed. pages 418–424; EU 3.4; LO 3.4D; EK 3.4D2; MPAC 2,4)