 Day 7 Review Series

MULTIPLE-CHOICE QUESTIONS

A calculator may not be used for the following questions.

D

1. Which of the following series is absolutely convergent?

   (A) $\sum_{k=0}^{\infty}(-1)^k \frac{k+3}{k+\sqrt{k}}$

   (B) $\sum_{k=0}^{\infty}(-1)^k \frac{3}{\sqrt{k}}$

   (C) $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{\sqrt{k}}{k+3}$

   (D) $\sum_{k=0}^{\infty}(-1)^{k+1} \frac{3}{k\sqrt{k}}$ $= \frac{3}{k^{3/2}}$

P series
$p > 1$
converges

B

2. The power series $\sum_{k=1}^{\infty}\frac{x^k}{k}$ converges for which values of x?

   (A) $-1 < x < 1$
   (B) $-1 \le x < 1$
   (C) $-1 \le x \le 1$
   (D) x is any real number.

D

3. Which of the following series is the power series expansion for $f(x) = x(\cos x - 1)$?

   (A) $x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots$
   (B) $-x^3 + x^5 - x^7 + \cdots$
   (C) $\frac{x^3}{2} - \frac{x^5}{24} + \frac{x^7}{720} - \cdots$
   (D) $-\frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

D

4. What are all values of a for which the series $\sum_{k=0}^{\infty}\left(\frac{5}{9-a}\right)^k$ converges?

   (A) $a < 4$
   (B) $4 < a < 14$
   (C) $a < 9$
   (D) $a < 4$ or $a > 14$

$\left|\frac{5}{9-a}\right| < 1$

$5 < |9-a|$

$5 < 9-a$ or $-5 > 9-a$

$-4 < -a$ $\quad -14 > -a$

$4 > a$ $\quad 14 < a$

② $\lim_{k\to\infty}\left|\frac{\frac{x^{k+1}}{k+1}}{\frac{x^k}{k}}\right| = \lim_{k\to\infty}\left|\frac{k}{x^k}\cdot\frac{x^{k+1}}{k+1}\right| = \lim_{k\to\infty}\frac{k}{k+1}|x| = |x|\lim_{k\to\infty}\frac{k}{k+1} = |x| < 1$

$-1 < x < 1$

check endpts $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ Con $\quad \sum_{k=1}^{\infty}\frac{1}{k}$ Div

B

5. The Maclaurin series for $f(x) = \frac{1}{1+x^2}$ is $\sum_{k=0}^{\infty}(-1)^k x^{2k}$. What is the Maclaurin series for $g(x) = \tan^{-1} x$?

   (A) $C + \sum_{k=0}^{\infty}(-1)^k(2k)x^{2k-1}$
   (B) $C + \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}$
   (C) $C + \sum_{k=0}^{\infty}(-1)^{k+1} \frac{x^{2k+1}}{2k}$
   (D) $C + \sum_{k=0}^{\infty}(-1)^{k-1} \frac{x^{2k-1}}{2k}$

$= \int\frac{1}{1+x^2} = \int\sum(-1)^k x^{2k}$

$= \sum(-1)^k \frac{x^{2k+1}}{2k+1} + C$

A

6. The Maclaurin series $\sum_{k=0}^{\infty}(-9)^k\frac{x^{4k}}{(2k)!}$ represents which function below?

   (A) $\cos(3x^2)$
   (B) $\sin(3x^2)$
   (C) $\cos(9x^4)$
   (D) e^{-3x^2}

$\cos(3x^2) = \sum\frac{(-1)^k(3x^2)^{2k}}{(2k)!}$

$= \sum\frac{(-1)^k 3^{2k} x^{4k}}{(2k)!}$

C

7. If the first five terms of the Taylor expansion for $f(x)$ about $x = 0$ are $3 - 7x + \frac{5}{2}x^2 + \frac{3}{4}x^3 - 6x^4$, then $f'''(0) =$

   (A) $\frac{1}{8}$
   (B) $\frac{3}{4}$
   (C) $\frac{9}{2}$
   (D) 6

$\frac{3}{4} = \frac{f'''(0)}{3!}$

$\frac{3(3\cdot2\cdot1)}{4} = f'''(0)$

D

8. Which of the following series diverge?

   I. $\sum_{k=0}^{\infty}\frac{k^{\frac{3}{2}}+1}{5k^2+7}$

   II. $\sum_{k=2}^{\infty}(-1)^k\frac{1}{\ln(k)}$ alt series Con

   III. $\sum_{k=0}^{\infty}(-1)^k\left(\frac{4}{3}\right)^k$ geo $r > 1$ Div

   (A) I only
   (B) II only
   (C) I and II only
   (D) I and III only

I. $\frac{1}{k^{1/2}} \to p<1$ Div

$\lim_{k\to\infty}\frac{\frac{k^{1/2}}{k^{3/2}+1}}{5k^2+7} = \lim_{k\to\infty}\frac{5k^2+7}{k^{3/2}+1}\cdot\frac{1}{k^{1/2}} = \lim_{k\to\infty}\frac{5k^2+7}{k^2+k^{1/2}}$

$= 5 \ne 0$ or ∞

∴ series behave the same LCT

A calculator may be used for the following questions.

9. The sixth degree term of the Taylor series expansion for $f(x) = e^{-\frac{1}{2}x^2}$ about $x = 0$ has coefficient

(A) $-\frac{1}{48}$

(B) $-\frac{1}{6}$

(C) $\frac{1}{720}$

(D) $-\frac{1}{4608}$

$\frac{\left(-\frac{1}{2}x^2\right)^3}{3!} = \frac{-1}{8} \cdot \frac{1}{6} x^6$

10. Let $T_n(x)$ represent the Taylor Polynomial of degree n about $x = 0$ for $f(x) = e^{-x}$. If $T_n(2)$ is used to approximate the value of $f(2) = e^{-2}$, then which of the following expressions is NOT less than $\frac{2^7}{7!}$?

$|f(2) - T_n(2)| < a_{n+1}(2)$

(A) $f(2) - T_6(2)$ $< a_7$

(B) $T_6(2) - f(2)$

(C) $f(2) - T_5(2)$ $= +$

(D) $T_5(2) - f(2)$ $= -$

$1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \text{error}$

*11. For function $f(x)$, $f(0) = 3$, $f'(0) = 2$, $f''(0) = 5$, and $f'''(0) = 4$. Using the Taylor series expansion for $f(x)$ about $x = 0$, the second degree estimate for $f'(0.1)$ is

(A) 2.500

(B) 2.520

(C) 3.120

(D) 3.225

$P_2 = 3 + 2x + \frac{5x^2}{2!} + \frac{4x^3}{3!}$

$P' = 2 + 5x + 2x^2$

$P'(.1)$

A calculator may not be used for the following questions.

12. The Maclaurin series $\sum_{k=0}^{\infty} (-5)^k \frac{x^{3k+2}}{k!}$ represents which expression below?

(A) $x^2 e^{-5x^3}$

(B) $-5e^{3x+2}$

(C) $x^2 \sin(-5x^3)$

(D) $-5\cos(x^3) + 2$

A

$e^x = \sum \frac{x^n}{n!}$

$e^{-5x^2} = \sum \frac{(-5x^2)^n}{n!} = \frac{(-5)^n x^{2n}}{n!}$

$x^2 \sum \frac{(-5)^n x^{2n}}{n!}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

13. If $f(x) = \sin(x^2)$, the first three terms of the Taylor series expansion about $x = 0$ for $f'(x)$ are

(A) $2x - x^3 + \frac{1}{2}x^5$

(B) $1 - \frac{1}{2}x^4 + \frac{1}{16}x^8$

(C) $2x - x^5 + \frac{1}{12}x^9$

(D) $x^2 - \frac{1}{6}x^6 + \frac{1}{120}x^{10}$

C

$f(x) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!}$

$f'(x) = 2x - x^5 + \frac{\cancel{10}x^9}{\cancel{5}\cdot 4\cdot 3\cdot \cancel{2}}$

14. Which of the following series is conditionally convergent?

(A) $\sum_{k=1}^{\infty} (-1)^k \frac{k^2+1}{k+4}$ Div nth term test

(B) $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{2}{k+1}$ LCT to 1/k, nonalt div

(C) $\sum_{k=1}^{\infty} (-1)^{k+1} \ln(k+1)$ Div nth term test

(D) $\sum_{k=1}^{\infty} (-1)^k \frac{k+3}{k^3+7}$ LCT to $1/k^2$ nonalt con

B

15. The radius of convergence of the power series $\sum_{k=0}^{\infty} \frac{x^{2k}}{k \cdot 4^k}$ is

(A) 0

(B) 1

(C) 2

(D) 4

FREE-RESPONSE QUESTION

A calculator may be used for this question.

Let $f(x)$ be a function that is differentiable for all x. The Taylor expansion for $f(x)$ about $x = 0$ is given by $T(x) = \sum_{k=0}^{\infty} (-1)^k \frac{(k+1)x^{3k}}{k!}$. The first four nonzero terms of $T(x)$ are given by $T_4(x) = 1 - 2x^3 + \frac{3x^6}{2!} - \frac{4x^9}{3!}$.

(a) Show that $T(x)$ converges for all x.

(b) Let $W(x)$ be the Taylor expansion for $x^2 f'(x)$ about $x = 0$. Find the general term for $W(x)$ and find $W_3(x)$.

(c) $f(0.5) \approx T_4(0.5)$ and $f(1) \approx T_4(1)$. Find the values of $T_4(0.5)$ and $T_4(1)$.

(d) Which value is smaller, $|f(0.5) - T_4(0.5)|$ or $|f(1) - T_4(1)|$? Give a reason for your answer.

15) $\lim_{n\to\infty} \left| \frac{x^{2(k+1)}}{(k+1)4^{k+1}} \cdot \frac{k \cdot 4^k}{x^{2k}} \right| = \left|\frac{x^2}{4}\right| \lim_{k\to\infty} \frac{k}{k+1} = \frac{x^2}{4} < 1$

$x^2 < 4$

$-2 < x < 2$

FREE-RESPONSE QUESTION

	Solution	Possible points
(a)	Using the Ratio Test for absolute convergence, $\lim_{k\to\infty}\left\|\dfrac{\frac{(k+2)x^{3(k+1)}}{(k+1)!}}{\frac{(k+1)x^{3k}}{k!}}\right\| < 1 \Rightarrow$ $\lim_{k\to\infty}\left\|\dfrac{(k+2)x^{3k+3}}{(k+1)!}\cdot\dfrac{k!}{(k+1)x^{3k}}\right\| < 1 \Rightarrow$ $\lim_{k\to\infty}\left\|\dfrac{k+2}{k+1}\cdot\dfrac{x^3}{k+1}\right\| < 1 \Rightarrow 1\cdot 0 < 1$ which is true for all x. Therefore $T(x)$ converges for all real numbers.	2: 1: correct use of Ratio Test for Absolute Convergence 1: answer
(b)	$f'(x) = -3\cdot 2x^2 + \dfrac{6\cdot 3x^5}{2!} - \dfrac{9\cdot 4x^8}{3!} + \cdots =$ $\sum_{k=1}^{\infty}(-1)^k\dfrac{3k(k+1)x^{3k-1}}{k!}$. Therefore $W(x) = x^2 f'(x) =$ $\sum_{k=1}^{\infty}(-1)^k\dfrac{3k(k+1)x^{3k+1}}{k!}$ and $W_3(x) = -6x^4 + 9x^7 - 6x^{10}$.	4: 1: expanded $f'(x)$ 1: general term of $f'(x)$ 1: general term of $W(x)$ 1: $W_3(x)$
(c)	Using direct substitution or a graphing calculator evaluation, $T_4(0.5) = 0.772$ and $T_4(1) = -0.167$.	1: both answers correct
(d)	Because $T(x)$ is an alternating series, each difference above is smaller than the fifth term of the expansion, $\dfrac{5x^{12}}{4!}$, and $\dfrac{5(0.5)^{12}}{4!} < \dfrac{5(1)^{12}}{4!}$ $(0.00005 < 0.20833)$. Therefore, $\lvert f(0.5) - T_4(0.5)\rvert$ is smaller. Alternatively, $\lvert f(0.5) - T_4(0.5)\rvert$ is smaller because 0.5 is closer than 1 to the center $x = 0$ of the interval of convergence. Therefore the series converges to $f(x)$ more quickly.	2: 1: answer 1: reason

(a), (b) (*Calculus for AP* 1st ed. pages 595–601; EU 4.2; LO 4.2; EK 4.2C2,4.2B5; MPAC 2,3)

(c), (d) (*Calculus for AP* 1st ed. pages 584–591; EU 4.2; LO 4.2A,4.2B; EK 4.2A2,4.2B4; MPAC 2,3)