Serions Review

① $f(x) = \frac{1}{x-1}$ $\quad f(2) = 1$

a) $f'(x) = -1(x-1)^{-2}$ $\quad f'(2) = -1$

$f''(x) = 2(x-1)^{-3}$ $\quad f''(2) = 2$

$f'''(x) = -6(x-1)^{-4}$ $\quad f'''(2) = -6$

$f(x) = 1 - (x-2) + \frac{2(x-2)^2}{2!} - \frac{6(x-2)^3}{3!}$

$$\boxed{= 1 - (x-2) + (x-2)^2 - (x-2)^3 + \ldots + (-1)^n(x-2)^n}$$

b) $g(x) = \ln|x-1| = \int g'(x)$

$g'(x) = \frac{1}{x-1}$

$\ln|x-1| = K + \sum_{n=0} \frac{(-1)^n (x-2)^{n+1}}{n+1}$

$\ln 1 = K + \sum 0$

$0 = K$

$$\boxed{\ln|x-1| = (x-2) - \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} - \frac{(x-2)^4}{4} + \ldots + \frac{(-1)^n (x-2)^{n+1}}{n+1}}$$

② $f(x) = e^{-2x^2}$

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$ $\quad \frac{(-1)^n (2)^n x^{2n}}{n!}$

$e^{-2x^2} = 1 + -2x^2 + \frac{(-2x^2)^2}{2!} + \frac{(-2x^2)^3}{3!} + \ldots + \frac{(-2x^2)^n}{n!}$

b) Ratio test $\lim_{n\to\infty} \left| \frac{(-2x^2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-2x^2)^n} \right| = \lim_{n\to\infty} \left| \frac{(-2x^2)}{n+1} \right| = 0 < 1$

$R = \infty$

Series converges everywhere $(-\infty, \infty)$

c) $|R_n(x)| \le |a_{n+1}|$ $\qquad e^{-2} = f(1)$

$|R_5(x)| \le \left| \frac{(2x^2)^3}{3!} \right|$

$R_5(1) < \frac{(2\cdot 1^2)^3}{3!} = \frac{8}{6} = \frac{4}{3}$

(3) a) $P_2(x) = 3 - 2(x-1) + \frac{2(x+1)^2}{2!}$

$f(.7) \approx 3 - 2(.7-1) + (.7-1)^2$

$\boxed{\approx 3.69}$

b) $P_3(x) = 3 - 2(x-1) + (x-1)^2 + \frac{4(x-1)^3}{3!}$

$f(1.2) \approx 3 - 2(1.2-1) + (1.2-1)^2 + \frac{2(1.2-1)^3}{3}$

$\boxed{\approx 2.645}$

c) $f'(x) \approx P_2(x) = -2 + 2(x-1) + \frac{2\cdot 3(x-1)^2}{3}$

$f'(1.2) \approx -2 + 2(1.2-1) + 2(1.2-1)^2$

$\boxed{\approx -1.52}$

(4)

n	$f^n(x)$	$f^n(1)$
0	$f(x)=e^{3x}$	e^3
1	$f'(x)=3e^{3x}$	$3e^3$
2	$f''(x)=9e^{3x}$	$9e^3$
3	$f'''(x)=27e^{3x}$	$27e^3$

$$e^{3x} = e^3 + 3e^3(x-1) + \frac{9e^3(x-1)^2}{2!} + \frac{27e^3(x-1)^3}{3!} + \dots + \frac{3^n e^3 (x-1)^n}{n!} + \dots$$

$$= e^3 \sum_{n=0}^{\infty} \frac{3^n (x-1)^n}{n!}$$

(5) $f(x) = 2(x-3)^{-2}$

$$\int f(x) = \frac{2(x-3)^{-1}}{-1} = \frac{-2}{x-3} = \frac{2/3}{1-\frac{x}{3}} = C + \sum_{n=0}^{\infty} \frac{2}{3}\left(\frac{x}{3}\right)^n = C + \sum_{n=0}^{\infty} \frac{2x^n}{3^{n+1}}$$

$\left|\frac{x}{3}\right| < 1$

$$\frac{d}{dx}\int f(x) = f(x) = \sum_{n=1}^{\infty} \frac{2n x^{n-1}}{3^{n+1}}, \quad -3 < x < 3$$

endpts

$\sum \frac{2n3^{n-1}}{3^{n+1}}$

$\sum \frac{2n}{9}$ Div

$\sum (-1)^{n-1} \frac{2n}{9}$ Div

both by n^{th} term test

(6) $$\lim_{n\to\infty} \left| \frac{(3x)^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(3x)^n} \right| = \lim_{n\to\infty} \left| \frac{3x}{(2n+2)(2n+1)} \right| = 0 < 1$$

$2n+2$

$R = \infty$

converges for all x $(-\infty, \infty)$

(7) $$\lim_{n\to\infty} \left| \frac{(x-2)^{n+2}}{(n+2)\,4^{n+2}} \cdot \frac{(n+1)\,4^{n+1}}{(x-2)^{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{4(n+2)} \right|$$

$$= \left| \frac{x-2}{4} \right| < 1$$

$|x-2| < 4$

$R = 4$

$(2-4, 2+4)$

$(-2, 6)$

$\boxed{[-2, 6)}$

$x=-2$ $\sum_{n=0} \frac{(-2-2)^{n+1}}{(n+1)4^{n+1}} = \sum_{n=0} \frac{(-4)^{n+1}}{4^{n+1}(n+1)}$

$= \sum_{n=0} \frac{(-1)^{n+1}}{n+1}$ alt. series conv.

$x=6$ $\sum_{n=0} \frac{(6-2)^{n+1}}{4^{n+1}(n+1)} = \sum \frac{4^{n+1}}{4^{n+1}(n+1)}$

$= \sum \frac{1}{n+1}$ div

DCT $\frac{1}{n}$

(8) $\sum_{n=0} \frac{(-1)^n x^{2n+1}}{2n+1}$

$\lim_{n \to \infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{2n+3} \cdot \frac{2n+1}{(-1)^n x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2(2n+1)}{(2n+2)} \right|$

$= |x| < 1$

$(-1, 1)$

$x=-1$ $\sum_{n=0} \frac{(-1)^n(-1)^{2n+1}}{2n+1} = \sum_{n=0} \frac{(-1)^{3n+1}}{2n+1}$ alt. Series

conv.

$\boxed{[-1, 1]}$

$x=1$ $\sum \frac{(-1)^n}{2n+1}$ alt series conv.