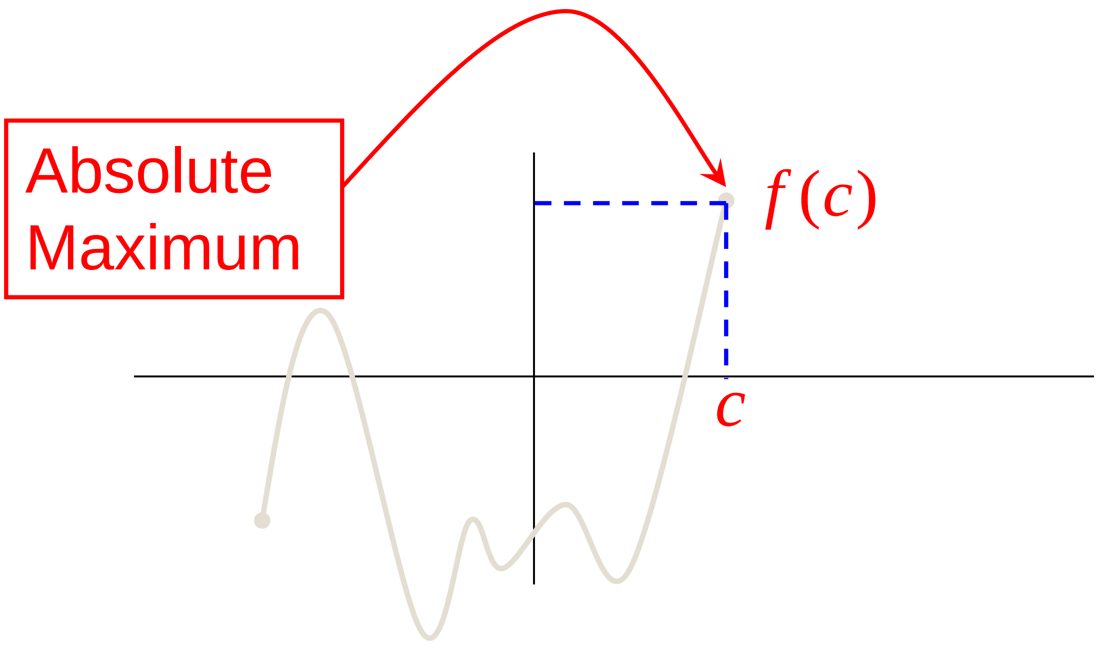
4.1 Maximum and Minimum Values

What is your definition of a maximum or minimum value on a graph?

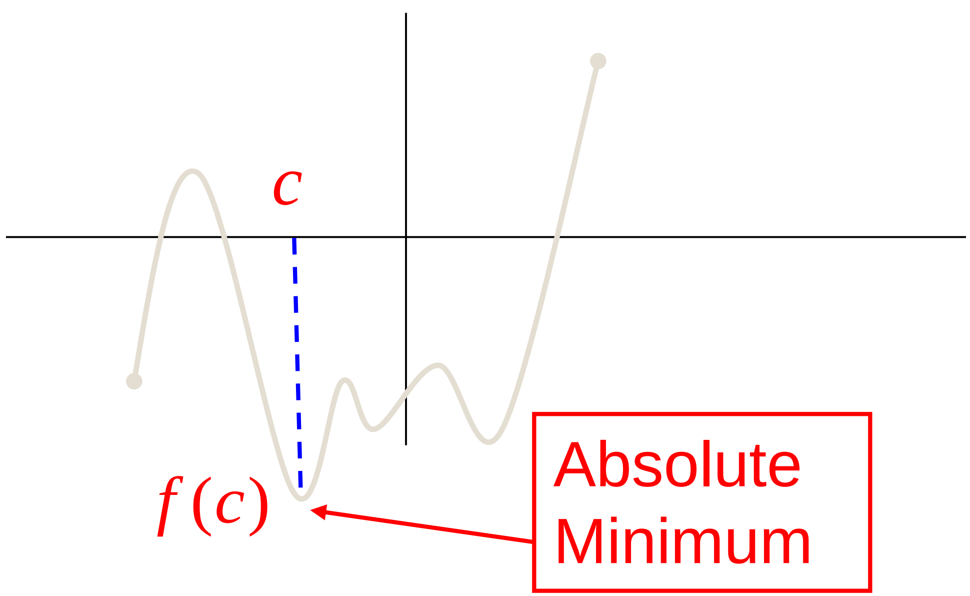
A function f has an ***absolute (global) maximum*** at $x = c$ if $f(c) \geq f(x)$ ***for all*** x in the domain D of f.

The number $f(c)$ is called the ***absolute maximum*** value of f in D



A function f has an ***absolute (global) minimum*** at $x = c$ if $f(c) \le f(x)$ ***for all*** x in the domain D of f.

The number $f(c)$ is called the ***absolute minimum*** value of f in D



Local Maximum and Minimum

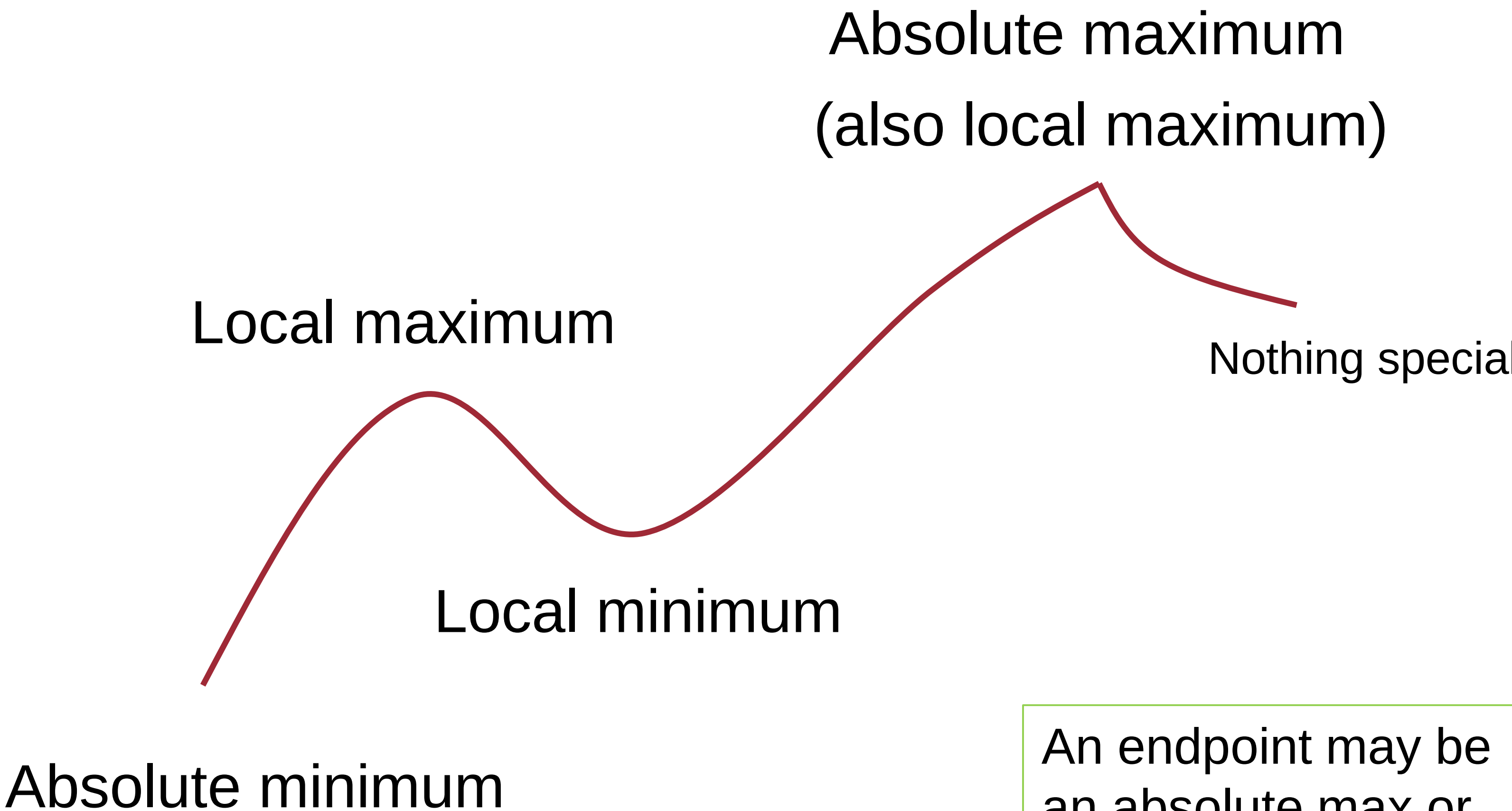
Definition:

- A function f has a **local maximum** (or **relative maximum**) at c if $f(c) \geq f(x)$ when x is near c.

- Similarly, f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c.

- The maximum and minimum values of f are called the **extreme values** of f.

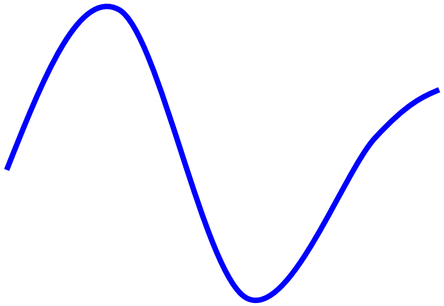
Example:



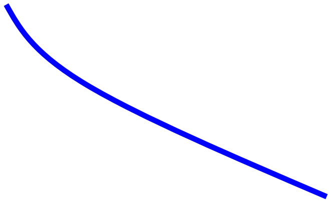
An endpoint may be an absolute max or min, but not a relative max or min.

Extreme Value Theorem:

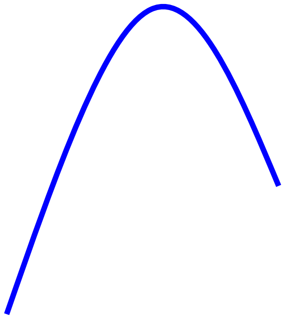
If f is continuous over a closed interval, then f has absolute maximum and minimum over that interval.



Maximum & minimum at interior points



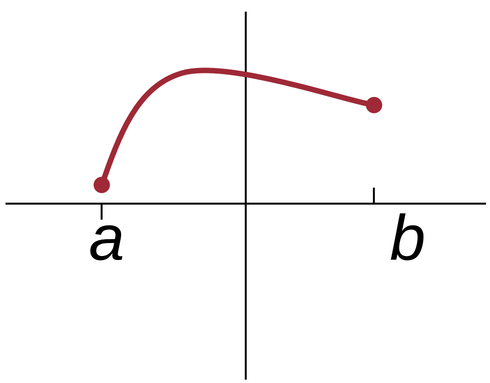
Maximum & minimum at endpoints



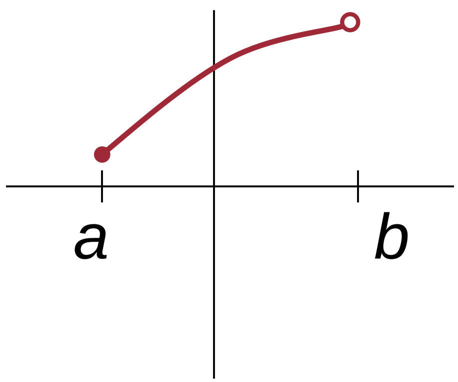
Maximum at interior point, minimum at endpoint

Extreme Value Theorem

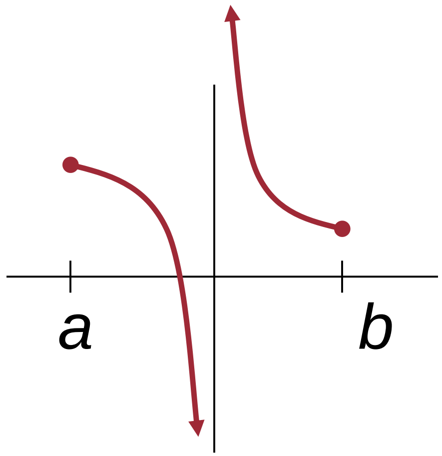
If a function *f* is continuous on a closed interval $[a, b]$, then *f* attains an absolute maximum and absolute minimum on $[a, b]$.



Attains max. and min.



Attains min. but no max.

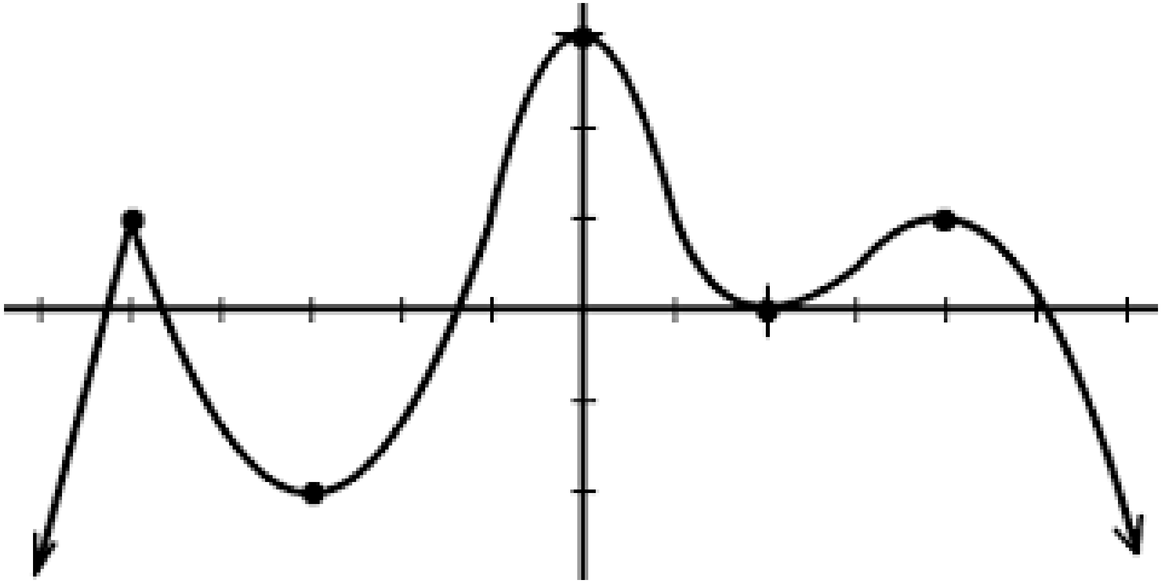


No min. and no max.

Open Interval

Not continuous

"Where" or "when" means x-value.
"What" means y-value.



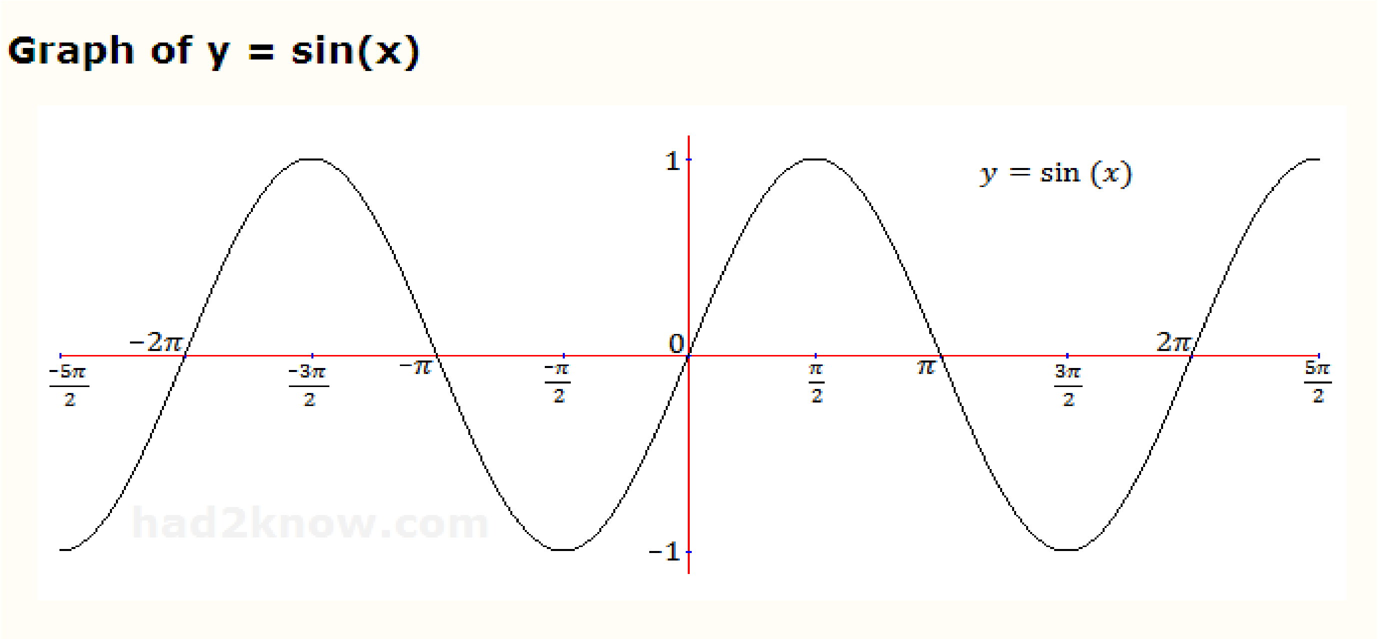
The absolute max value is 3, which occurs at x = 0.

The three local max values are 3 at x = 0 and 1 at x = -5 and 4.

The two local min values are -2 at x = -3 and 0 at x = 2.

There is no absolute min value.

There can be only one absolute min or max value, but it can occur at many locations.



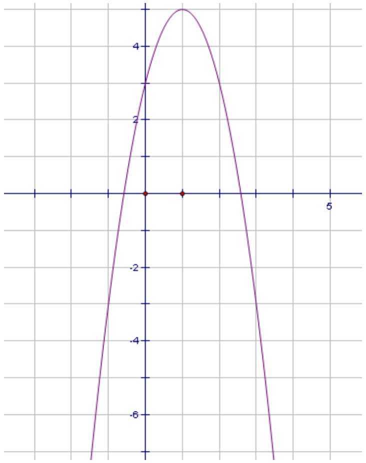
Absolute max of 1 occurs at an infinite number of x-values.

Let's look at $y=x^2 - 6x$ and $y = -2x^2 + 4x + 3$ on the calculator.

Graph: 

Use calculator to find the min. It occurs at ?

X = 3



Use calculator to find the max. It occurs at ?

X = 1

Use calculator to find y'(1) = -4
y'(3) = 0
y'(5) = 4

2nd Trace, 6: dy/dx

Use calculator to find y'(-1) = 8
y'(1) = 0
y'(3) = -8

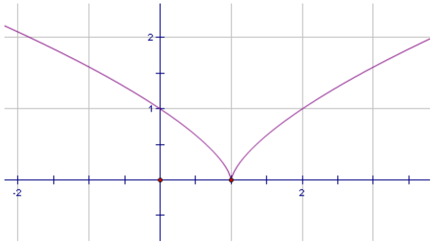
What can you hypothesize about the derivative of a function at a point that is an extrema?

Derivative equals zero.

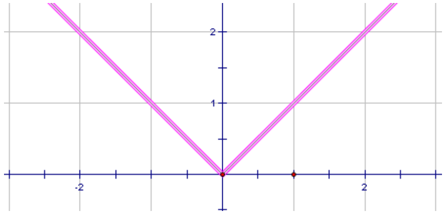
Sketch the graphs without a calculator.

$$f(x) = (x-1)^{2/3}$$

-radical
-even
-shift right 1



$$g(x) = |x|$$



f'(1) = ? Does not exist

Extrema? Min at x = 1

g'(0) = ? Does not exist

Extrema? Min at x = 0

Conclusion?

Extrema can occur at locations where the derivative does not exist.

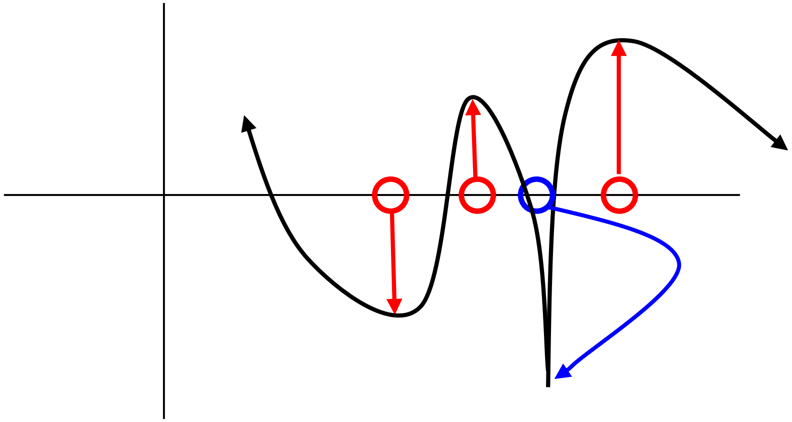
Critical Points of f

A ***critical number*** of a function f is a number c ***in the domain of*** f such that

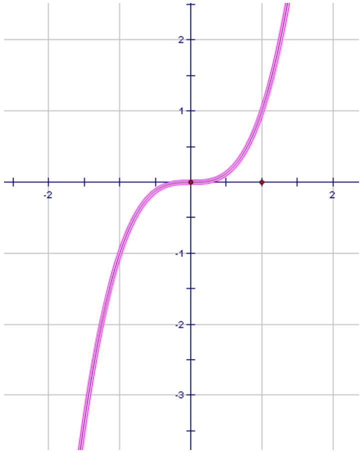
a. $f'(c) = 0$

or

b. $f'(c)$ does not exist



Sketch $f(x) = x^3$.



$f'(0) = ?$ 0

But the function does not have any extrema.

Find $f'(-2)$ and $f'(2)$ Both equal 12

The slopes stay positive.

Conclusion?

Graph needs to change direction to create an extreme value.

Increasing/Decreasing/Constant

If $f'(x) > 0$ for each value of x in an interval (a,b), then f is increasing on (a,b).

If $f'(x) < 0$ for each value of x in an interval (a,b), then f is decreasing on (a,b).

If $f'(x) = 0$ for each value of x in an interval (a,b), then f is constant on (a,b).

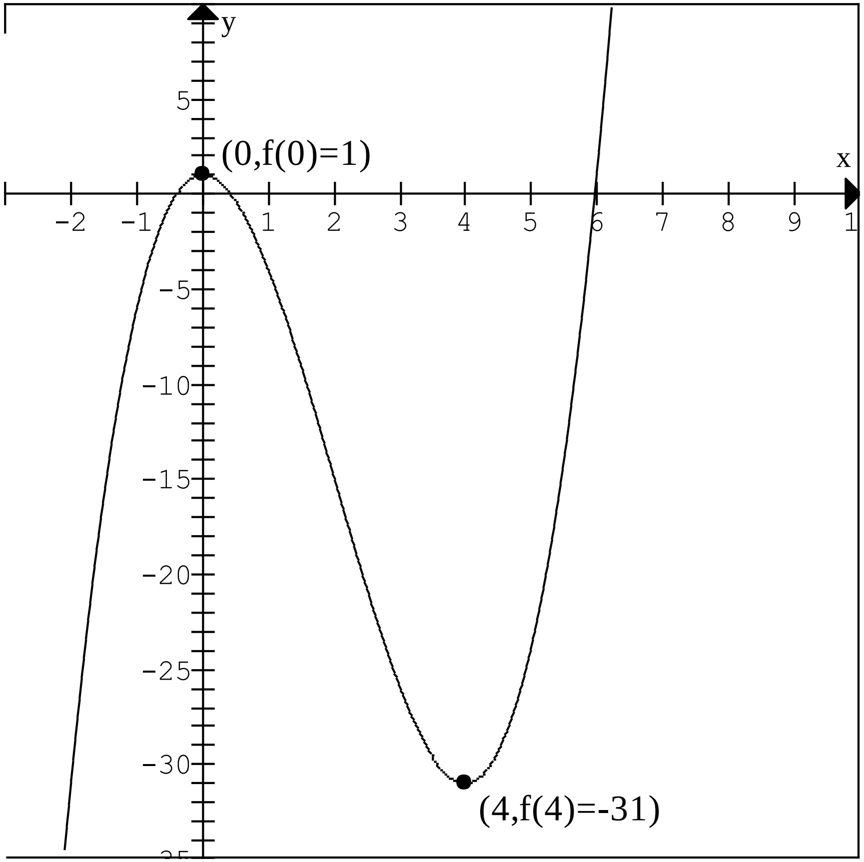
1^{st} Derivative Test

- Find the critical points
  - Set derivative = 0
  - Where is derivative undefined
- Analyze sign behavior
  - Create a sign chart with the critical points
  - Test #s and see if you get + or - results
- Interpret the results
  - Where is function increasing/decreasing
  - Where are extrema
  - What type of extrema

The First Derivative Test

Find all the relative extrema of $f(x) = x^3 - 6x^2 + 1.$

Graph of $f(x) = x^3 - 6x^2 + 1.$



Another Example

Find all the relative extrema of $f(x) = \sqrt[3]{x^3 - 3x}.$

$$f'(x) = \frac{x^2 - 1}{\sqrt[3]{\left(x^3 - 3x\right)^2}}$$

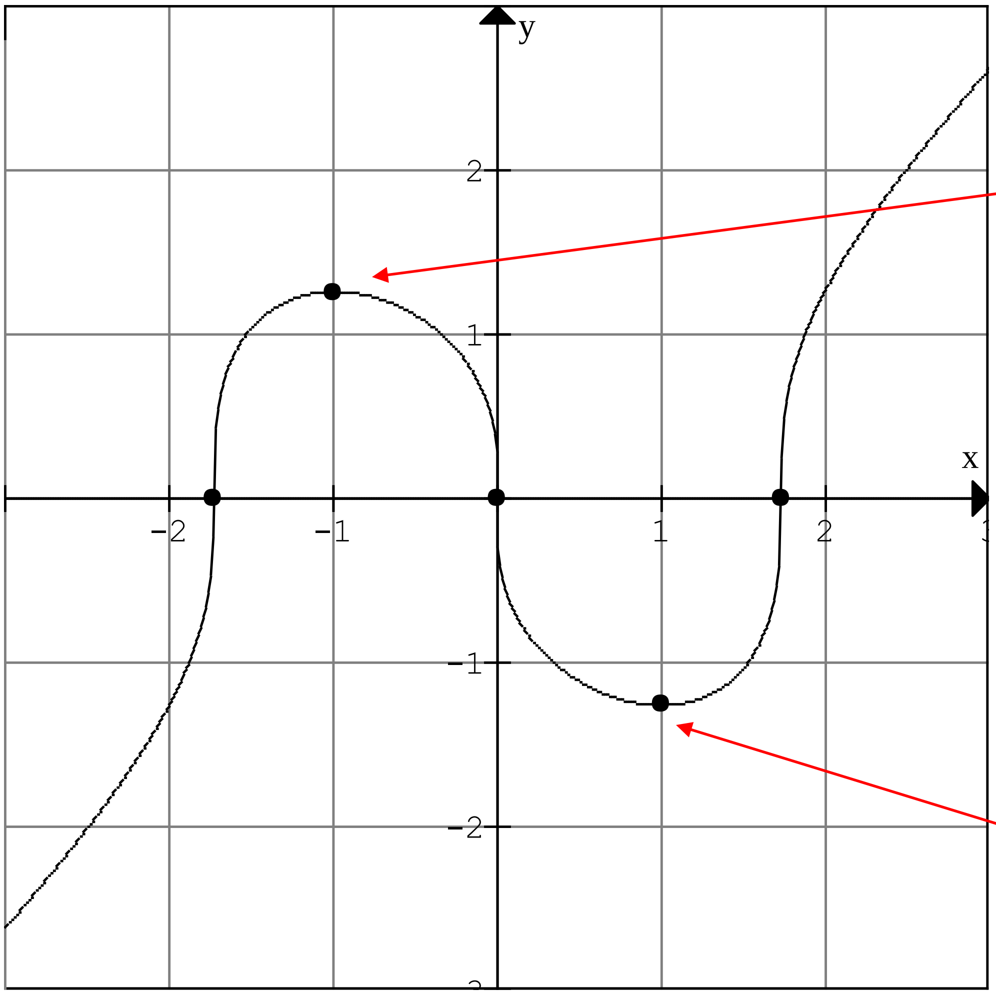
f'(x) = 0: $x = \pm 1$

f'(x) is undefined: $x = 0, \pm\sqrt{3}$

f’(x) = 0: $x = \pm 1$

F’(x) is undefined: $x = 0, \pm\sqrt{3}$

Graph of $f(x) = \sqrt[3]{x^3 - 3x}.$



Local max. $f(-1) = \sqrt[3]{2}$

Local min. $f(1) = -\sqrt[3]{2}$

Another Example

Find all the relative extrema of $y = x^2 e^x$

On a closed interval you must consider endpoints as well as critical numbers.

- Find the absolute min and max of $f(x) = x^2 - 10x + 27$ on the interval $[1, 6]$.

Find all local and absolute extrema for

$$y = \frac{x^3}{3} - 3x^2 + 8x \qquad 0 \le x \le 10$$

Find all local and absolute extrema for

$$y = \frac{1}{x} + \ln x \qquad \tfrac{1}{2} \le x \le 4$$

Find all local and absolute extrema for

$y = x^{2/3}$ $\qquad$ $-1 \leq x \leq 8$