Section 4.2 Mean Value Theorem

Think about it:

- You are going visit your Uncle, who is a highway patrolman, in Ashville. You call him to say you're leaving Fuquay at noon and arrive in Ashville, 224 miles away, at 3 pm. Before he even says hello, he hands you a ticket for speeding. How did he know?

At some point during your drive, your instantaneous velocity was equal to your average velocity.

Let’s look at this graphically.

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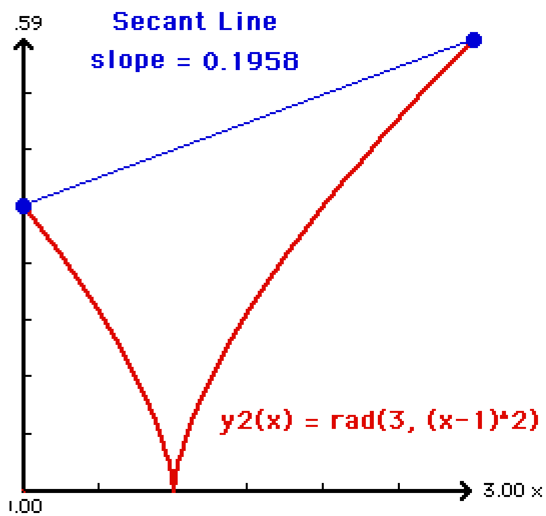
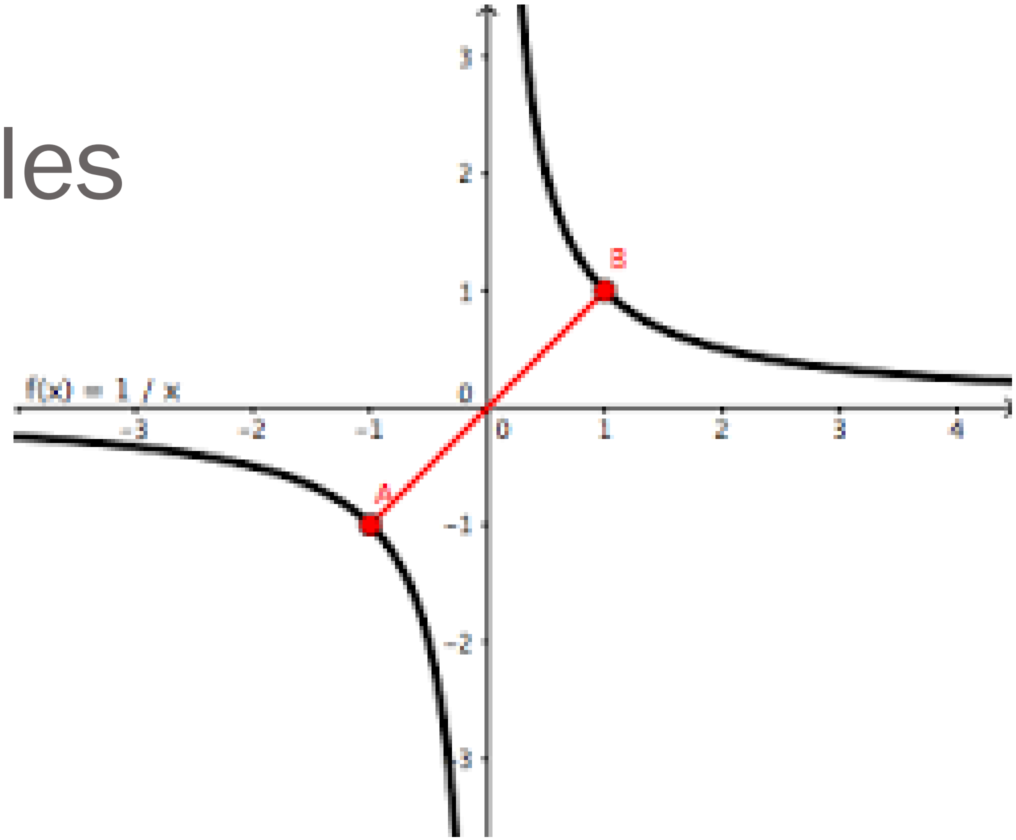
- Sketchpad-MVT
- Counterexample
- Counterexample 2

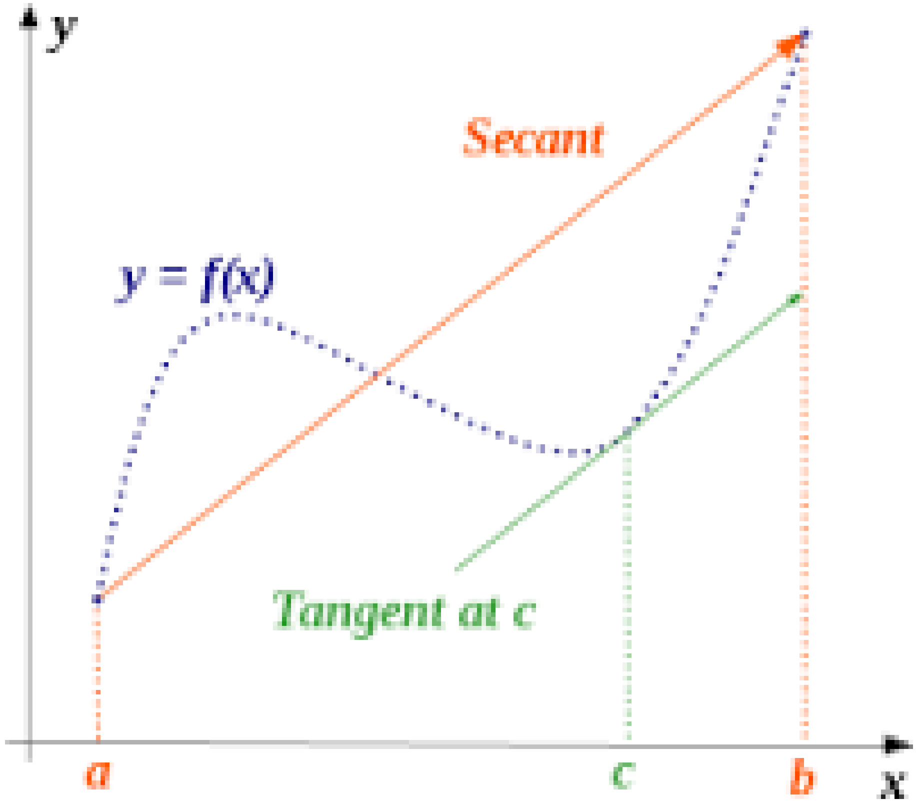
Counterexamples





What must be true about the function to insure there is a point where the instantaneous rate of change (IROC) equals the average rate of change (AROC)?

- In order for there to be a point on a given interval where the IROC is equal to the AROC, the function must be continuous on the closed interval [a,b] and differentiable on the open interval (a,b).



Example

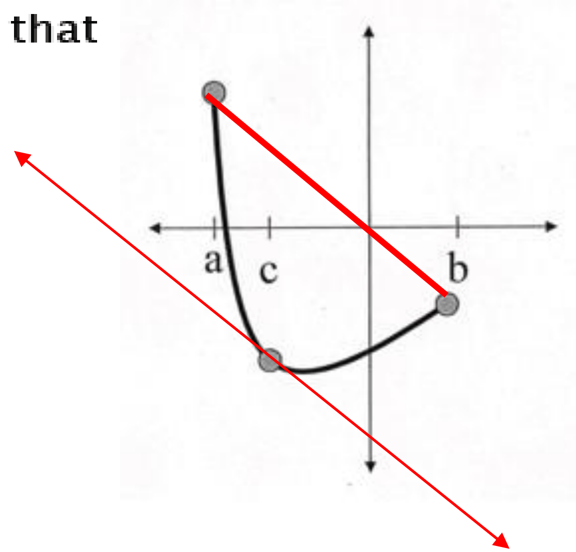
- Given the function $y = x^2$, where on the interval [-1,4] does the IROC equal the AROC?

Mean Value Theorem

If f is continuous on [a,b] and differentiable on (a,b)

Then there exists a number, c, in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Slope of tangent = Slope of secant

Instantaneous rate of change = Average rate of change

IROC = AROC

You try:

1. Find the value of c on the interval [-1, 2] for the equation $y = x^3 - 1$, such that the IROC is equal to the AROC.

2. Find the x-value(s) guaranteed to exist by the mean value theorem on [-3,1] for the equation $y = x^3 + \frac{5}{2}x^2 - 2x$

Review the IVT

Intermediate Value Theorem: If a function is continuous on [a,b], then the function takes on all values between f(a) and f(b).

Function *f* is continuous on [0,4]. Given the following table of values, what is the minimum number of times f(x) = 8? Justify.

X	0	1	2	3	4
Y	10	6	2	7	9

Because the function is continuous, f(x) = 8 at least 2 times. Once on the interval (0,1) and once on the interval (3,4) by the IVT.

Example

Function *f* is continuous on [0,10] and differentiable on (0,10). Given the following data, what is the minimum number of extrema that must exist on the interval (0,10)? Justify.

X	0	2	4	6	8	10
f	5	7	5	4	2	5

There are 2 extrema on the interval (0,10).
The average slope on the interval [0,4] is zero. Since the function is continuous and differentiable, the MVT guarantees f'(x) = 0 at a point on the interval. The function also changes from increasing to decreasing on this interval so a maximum exists.
The same reasoning guarantees an extrema on the interval [4,10].

Example

Function *f* is continuous on [6,10] and differentiable on (6,10). Given the following data, what is the minimum number of extrema that must exist on the interval (6,10)? Justify.

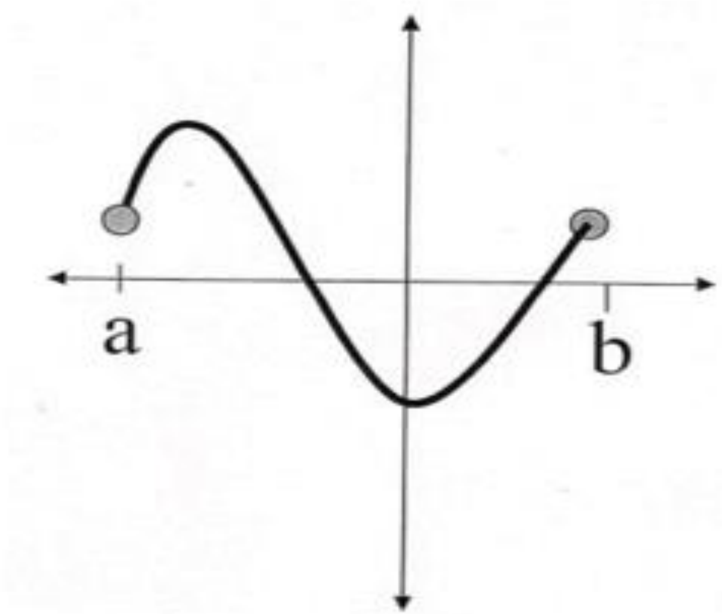
X	6	8	10
f	4	2	5

There is one extrema on the interval (6,10).
Because the function is continuous, the IVT says f(x) = 3 somewhere on the interval (6,8) and (8, 10).

The average slope between those points is zero. Because the function is continuous and differentiable the MVT says there is a location on the interval between those two points where f'(x) = 0. And because the function decreases then increases there must be an extrema.

Rolle's Theorem

- Let f be continuous on [a,b] and differentiable on (a,b). If f(a) = f(b), then there is at least one value c in (a,b) such that f'(c) = 0.



This is a special instance of the Mean Value Theorem.

Examples: Rolle's Theorem

1. Explain why the conclusion to Rolle's Theorem is not guaranteed for the function $f(x) = x / (x - 3)$ on the interval [1, 6].

2. Verify that the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers *c* that satisfy the conclusion of Rolle's Theorem.

$$f(x) = x\sqrt{x+6};\ [-6, 0]$$