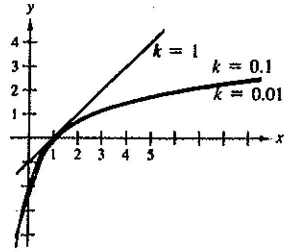
61. $f(x) = \dfrac{x^k - 1}{k}$

$k = 1, \qquad f(x) = x - 1$

$k = 0.1, \qquad f(x) = \dfrac{x^{0.1} - 1}{0.1} = 10(x^{0.1} - 1)$

$k = 0.01, \qquad f(x) = \dfrac{x^{0.01} - 1}{0.01} = 100(x^{0.01} - 1)$

$\lim\limits_{k \to 0} \dfrac{x^k - 1}{k} = \lim\limits_{k \to 0} \dfrac{x^k(\ln x)}{1} = \ln x$



Section 8.8 Improper Integrals

lim a→0+

1. $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_0^4 = 4$

2. $\displaystyle\int_3^4 \frac{1}{\sqrt{x-3}}\,dx = 2\sqrt{x-3}\Big]_3^4 = 2$

3. $\displaystyle\int_0^2 \frac{1}{(x-1)^{2/3}}\,dx = \int_0^1 \frac{1}{(x-1)^{2/3}} + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx = \Big[3(x-1)^{1/3}\Big]_0^1 + \Big[3(x-1)^{1/3}\Big]_1^2 = 6$

4. $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx = \left[\frac{-1}{x-1}\right]_0^1 + \left[\frac{-1}{x-1}\right]_1^2$

$\displaystyle = \left(\frac{-1}{0^-} - 1\right) + \left(-1 + \frac{1}{0^+}\right)$

$= (\infty - 1) + (-1 + \infty) = \infty$ diverges.

5. $\displaystyle\int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty = 0 + 1 = 1$

6. $\displaystyle\int_{-\infty}^0 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big]_{-\infty}^0 = \frac{1}{2} - 0 = \frac{1}{2}$

7. $\displaystyle\int_{-\infty}^0 xe^{-2x}\,dx = \frac{1}{4}\int_{-\infty}^0 (-2x)e^{-2x}(-2)\,dx = \frac{1}{4}\Big[(-2x-1)e^{-2x}\Big]_{-\infty}^0 = -\infty$ diverges.

8. $\displaystyle\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}(x+1)\Big]_0^b = \lim_{b\to\infty}[1 - e^{-b}(b+1)] = 1$ since $\displaystyle\lim_{b\to\infty}\left(\frac{b+1}{e^x}\right) = 0$ by L'Hôpital's Rule.

9. $\displaystyle\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}(x^2+2x+2)\Big]_0^b = \lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b} + 2\right) = 2$ since $\displaystyle\lim_{b\to\infty}\left(-\frac{b^2+2b+2}{e^b}\right) = 0$ by L'Hôpital's Rule.

10. $\displaystyle\int_0^\infty (x-1)e^{-x}\,dx = \lim_{b\to\infty}\Big[-xe^{-x}\Big]_0^b = \lim_{b\to\infty}\left(\frac{-b}{e^b} + 0\right) = 0$ by L'Hôpital's Rule.

11. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \frac{-1}{x}\Big]_1^\infty = 1$

12. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big]_1^\infty = \infty - 2$ diverges.

13. $\displaystyle\int_0^\infty e^{-x}\cos x\,dx = \frac{1}{2}\Big[e^{-x}(-\cos x + \sin x)\Big]_0^\infty = \frac{1}{2}[0 - (-1)] = \frac{1}{2}$

14. $\displaystyle\int_0^\infty e^{-ax}\sin bx\,dx = \left[\frac{e^{-ax}(-a\sin bx - b\cos bx)}{a^2+b^2}\right]_0^\infty = 0 - \frac{-b}{a^2+b^2} = \frac{b}{a^2+b^2}$

15. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = 2\int_0^\infty \frac{1}{1+x^2}\,dx = \Big[2\arctan x\Big]_0^\infty = 2\left(\frac{\pi}{2}\right) = \pi$

35. For $n = 1$ we have $\int_0^\infty xe^{-x}\,dx = 1$ (see Exercise 8). Assume $\int_0^\infty x^n e^{-x}\,dx$ converges, then for $n+1$ we have $\int_0^\infty x^{n+1}e^{-x}\,dx$. Using integration by parts, $u = x^{n+1}$, $dv = e^{-x}\,dx$, $du = (n+1)x^n\,dx$, $v = -e^{-x}$,

$$\int_0^\infty x^{n+1}e^{-x}\,dx = \Big[-x^{n+1}e^{-x}\Big]_0^\infty + (n+1)\int_0^\infty x^n e^{-x}\,dx = (n+1)\int_0^\infty x^n e^{-x}\,dx \text{ which converges.}$$

36. (a) Assume $\displaystyle\int_a^\infty g(x)\,dx = L$.

$$0 \le \int_a^\infty f(x)\,dx < \int_a^\infty g(x)\,dx = L$$

Therefore, $\displaystyle\int_a^\infty f(x)\,dx$ converges.

(b) Assume $\displaystyle\int_a^\infty f(x)\,dx = \infty$.

$$\int_a^\infty g(x)\,dx \ge \int_a^\infty f(x)\,dx = \infty$$

Therefore, $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

37. $\displaystyle\int_0^1 \frac{1}{x^3}\,dx$ diverges.
(See Exercise 34, $p = 3 \not< 1$.)

38. $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \frac{1}{1-(1/3)} = \frac{3}{2}$ converges .
(See Exercise 34, $p = 1/3$.)

39. $\displaystyle\int_1^\infty \frac{1}{x^3}\,dx = \frac{1}{3-1} = \frac{1}{2}$ converges.
(See Exercise 33, $p = 3$.)

40. $\displaystyle\int_0^\infty x^4 e^{-x}\,dx$ converges.
(See Exercise 35.)

41. Since $\dfrac{1}{x^2+5} \le \dfrac{1}{x^2}$ and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges by Exercise 33, $\displaystyle\int_1^\infty \frac{1}{x^2+5}\,dx$ converges.

42. Since $\dfrac{1}{\sqrt{x-1}} \ge \dfrac{1}{x}$ and $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 33, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x-1}}\,dx$ diverges.

43. Since $\dfrac{1}{\sqrt[3]{x(x-1)}} \ge \dfrac{1}{\sqrt[3]{x}}$ and $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x}}\,dx$ diverges by Exercise 33, $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x(x-1)}}\,dx$ diverges.

44. Since $\dfrac{1}{\sqrt{x}(1+x)} \le \dfrac{1}{x^{3/2}}$ and $\displaystyle\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges by Exercise 33, $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}(1+x)}\,dx$ converges.

45. Since $e^{-x^2} \le e^{-x}$ on $[0, \infty)$ and $\displaystyle\int_0^\infty e^{-x}\,dx$ converges (see Exercise 5), $\displaystyle\int_0^\infty e^{-x^2}\,dx$ converges.

46. $\dfrac{1}{\sqrt{x}\ln x} \ge \dfrac{1}{x}$ since $\sqrt{x} \ge \ln x$ on $[2, \infty)$. Since $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ diverges by Exercise 33, $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}\ln x}\,dx$ diverges.

47. (a) $\displaystyle A = \int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^\infty = 1$

(b) **Disc**

$$V = \pi\int_1^\infty \frac{1}{x^4}\,dx = \frac{\pi}{3}$$

(c) **Shell**

$$V = 2\pi\int_1^\infty x\left(\frac{1}{x^2}\right)dx = 2\pi(\ln x)\Big]_1^\infty = \infty$$

diverges.

48. (a) $\displaystyle A = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big]_0^\infty = 0 - (-1) = 1$

(b) **Shell**

$$V = 2\pi\int_0^\infty xe^{-x}\,dx = 2\pi\Big[-e^{-x}(x+1)\Big]_0^\infty = 2\pi$$