Section 9.2 Series

A **sequence** can be thought of as a list of numbers:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

If we try to add the terms of an infinite sequence we get an expression of the Form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n$$

partial sums

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4$

and, in general, $S_n = a_1 + a_2 + a_3 + \cdots + a_n$

Sequence of Partial Sums: $S_1, S_2, S_3, \ldots .. S_n$

The sum of an infinite series is the limit of its sequence of partial sums.

Example:

$1 - 1 + 1 - 1 + 1 - \ldots$ Converge or Diverge? (Does it have a finite sum?)

$S_1 = 1$

$S_2 = 1 - 1 = 0$

$S_3 = 1 - 1 + 1 = 1$

$S_4 = 1 - 1 + 1 - 1 = 0$

$S_n = \{ 1, 0, 1, 0, \ldots \}$

The sequence has no limit, so the series has no sum. The series diverges.

Example …

Converge or Diverge?

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots \quad = 1$$

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{8} + \frac{1}{16} = \frac{15}{16}$$

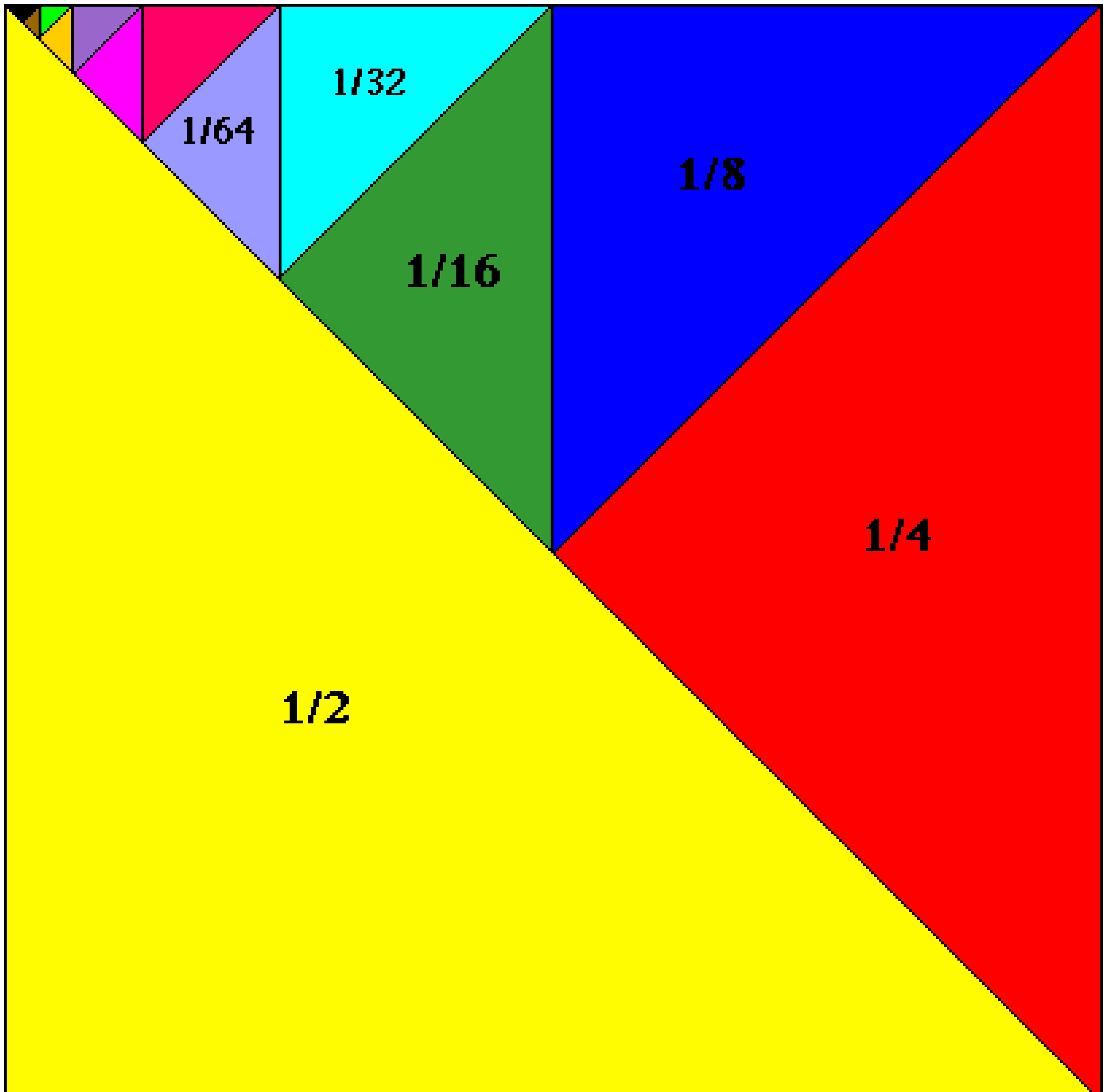
Pattern?

$$s_n = \frac{2^n - 1}{2^n}$$

$$\lim_{n\to\infty} \frac{2^n - 1}{2^n} = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1$$

NOTE: A general expression for s_n is usually difficult to determine.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^n} + \ldots = 1$$



Since finding a formula for the terms of the sequence of partial sums of an infinite series is sometimes difficult to find, we have several tests to determine the convergence of a series.

First let's see what the terms of the sequence tells about the convergence of the series.

Does the sequence $a_n = 2^n$ Converge or Diverge?

Does the series $\sum_{n=0}^{\infty} 2^n$ Converge or Diverge?

Does the sequence $a_n = \frac{n!}{2n!+1}$ Converge or Diverge?

Does the series $\sum_{n=0}^{\infty} \frac{n!}{2n!+1}$ Con. or Div.?

Does the sequence $a_n = \frac{1}{n}$ Converge or Diverge?

Does the series $\sum_{n=1}^{\infty} \frac{1}{n}$ Con. or Div.?

Nth term test for divergence:

Given $\sum a_n$, If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

(If the limit does equal zero, we do not know if the series converges or diverges)

Properties of Series

$$\sum ca_n = c\sum a_n$$

$$\sum \left(a_n \pm b_n\right) = \sum a_n \pm \sum b_n$$

Telescopic Series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

Converge or Diverge?

$$\sum_{n=3}^{\infty} \frac{1}{n^2 - 4}$$

Converge or Diverge?

Geometric Series

First term multiplied by *r*

Third term multiplied by *r*

$$\sum_{n=0}^{\infty} a \cdot r^n = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + a \cdot r^4 + \ldots$$

Second term multiplied by *r*

Note that in this case we start counting from zero. Technically it doesn't matter, but we have to be careful because the formula we will use starts always at n=0.

$$S_N = \sum_{n=0}^{N} a \cdot r^n$$

This is a geometric series in which you are multiplying by r to get each additional term

$$S_N = a + a \cdot r + a \cdot r^2 + a \cdot r^3 + \ldots + a \cdot r^N \qquad (1)$$

If we multiply both sides by *r* we get

$$r \cdot S_N = a \cdot r + a \cdot r^2 + a \cdot r^3 + \ldots + a \cdot r^N + a \cdot r^{N+1} \qquad (2)$$

If we subtract (2) from (1), we get

$$S_N - r \cdot S_N = a - a \cdot r^{N+1}$$

$$S_N(1-r) = \left(a - ar^{N+1}\right)$$

$$S_N = \frac{\left(a - ar^{N+1}\right)}{1-r}$$

This is the formula to generate the terms of the sequence of partial sums.

$$S_N = \frac{a - ar^{N+1}}{1-r}$$

So the limit of S_n as n approaches infinity gives you the sum of the infinite series.

$$If\,|r| > 1, \lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r} = \frac{a-\infty}{1-r} = \infty$$

$$If\,|r| < 1, \lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r} = \frac{a - a\cdot 0}{1-r} = \frac{a}{1-r}$$

Therefore, if you recognize a series is geometric, i.e. $\sum_{n=0}^{\infty} a\cdot r^n$

If $|r| \geq 1$, the series diverges.

If $|r| < 1$, the series converges to $\frac{a}{1-r}$, where a is the first term and r is the common ratio.

Examples: Converge or Diverge?

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=0}^{\infty} \frac{3}{2^n}$$

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$$

You can use series to change a repeating decimal to a fraction.

$$0.252525252525\ldots$$

$$=0.25 + 0.0025 + 0.000025 + \ldots$$

a =

r =

Sum =