Section 9.5 Alternating Series

Series with Positive Terms

Recall that series in which all the terms are positive have an especially simple structure when it comes to convergence.

Because each term that is added is positive, **the sequence of partial sums is increasing.**

So one of two things happens:

1. The partial sums stay bounded and the series converges,

OR

2. The partial sums go off to infinity and the series diverges.

What happens when a Series has some terms that are negative?

There are several Possibilities:

- All the terms are negative.

$$\sum_{n=0}^{\infty} -\frac{3}{n^8}$$

- Finitely many terms are negative.

$$-1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$$

- Infinitely many negative terms and infinitely many positive terms.

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots$$

$$-1-\frac{1}{2}+\frac{1}{4}+\frac{1}{8}-\frac{1}{16}-\frac{1}{32}+\cdots$$

Alternating Series

The simplest series that contain both positive and negative terms is an **alternating series,** whose terms alternate in sign. (involves -1 raised to a power)

For example, the geometric series

$$\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$$

$$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$$

is an *alternating geometric series* with $r = -\frac{1}{2}$.

Alternating series occur in two ways: either the odd terms are negative or the even terms are negative.

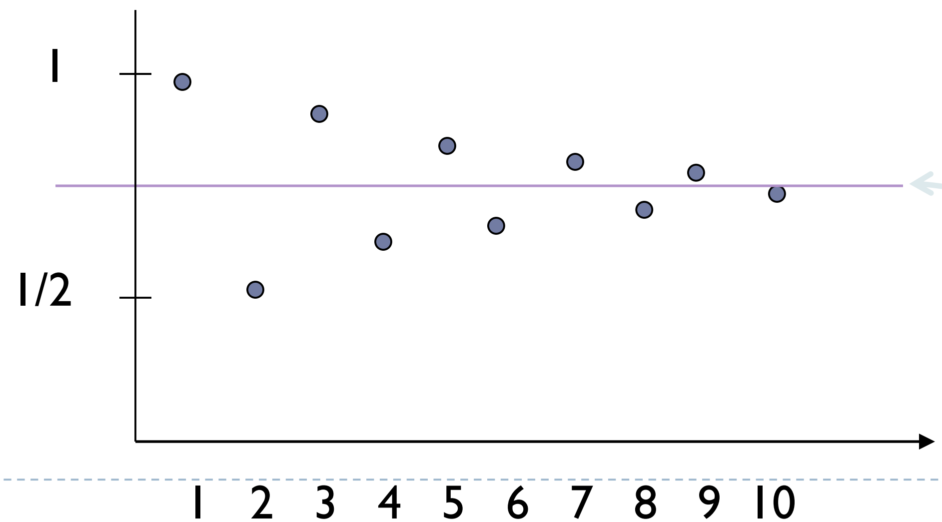
One of the most important Examples is:

The Alternating Harmonic Series

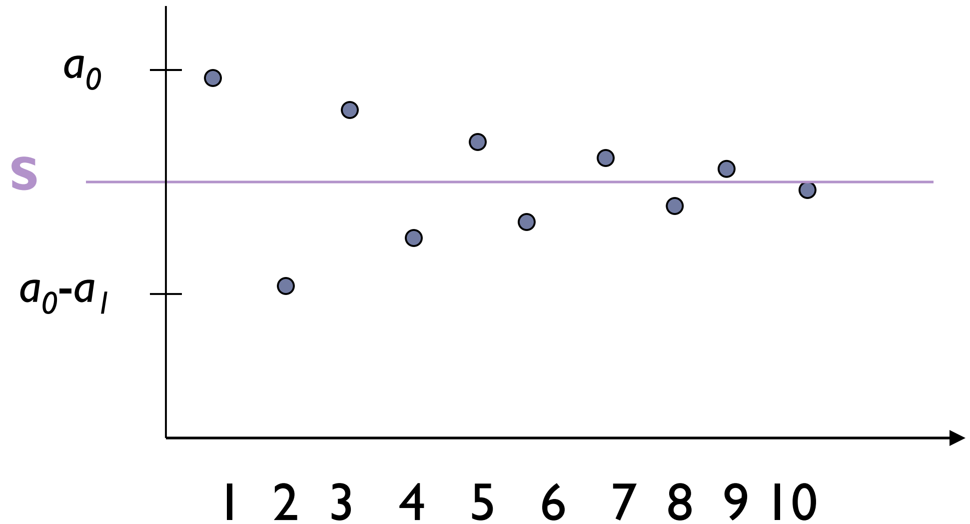
$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots$$

The alternating harmonic series seems to converge to a point about here

In order to determine whether the series converges, we need to examine the partial sums of the series.



The Idea behind the AST



There were two things that made this picture "go."

- The size (in absolute value) of the terms was decreasing.
- The terms were going to zero.

Alternating Series Test

Recall $\lim_{k\to\infty} a_k = 0$ does not guarantee convergence of the series

In case of <u>alternating</u> series $\sum_{k=1}^{\infty}(-1)^{k-1}a_k$

The series must converge if

- $\lim_{k\to\infty} a_k = 0$ and
- $\{ a_k \}$ is a decreasing sequence (that is $a_{k+1} \leq a_k$ for all k)

Alternating Series Test

- Start out by calculating $\lim_{k \to \infty} a_k$
- If limit ≠ 0, you know it diverges
- If the limit = 0
  - Proceed to verify { a_k } is a decreasing sequence
- Try it …

$$\frac{2}{1} - \frac{1}{1} + \frac{2}{2} - \frac{1}{2} + \frac{2}{3} - \frac{1}{3} + \frac{2}{4} - \frac{1}{4} + \ldots + \frac{2}{n} - \frac{1}{\mathrm{n}}$$

$$\lim_{n \to \infty} \left(\frac{2}{n} - \frac{1}{n} \right) = 0$$

Terms are approaching zero (condition 1), but each successive term is not necessarily less than the one before it (condition 2). Therefore, the alternating series test is inconclusive.

Example 2 – *Using the Alternating Series Test*

Determine the convergence or divergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Solution:

Note that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0.$

So, the first condition of Alternating Series Test is satisfied.

Also note that the second condition is satisfied because

$$a_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = a_n$$

for all *n*.

So, applying the Alternating Series Test, you can conclude that the series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

Conditional Convergence

- It is still possible that even though $\sum |a_k|$ diverges …
  - $\sum a_k$ can <u>still</u> converge
- This is called <u>conditional</u> convergence
- Example – consider $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ vs. $\sum_{k=1}^{\infty} \frac{1}{k}$

Absolute and Conditional Convergence

The **alternating harmonic series**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges by the Alternating Series Test. Yet the harmonic series diverges. This type of convergence is called **conditional.**

Definitions of Absolute and Conditional Convergence

1. The series $\Sigma\, a_n$ is **absolutely convergent** when $\Sigma\, |a_n|$ converges.
2. The series $\Sigma\, a_n$ is **conditionally convergent** when $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

Absolute Convergence

- Consider a series $\sum a_k$ where the general terms vary in sign
  - The alternation of the signs may or may not be any regular pattern
- If $\sum |a_k|$ converges and so does $\sum a_k$
- This is called <u>absolute</u> convergence

Absolutely!

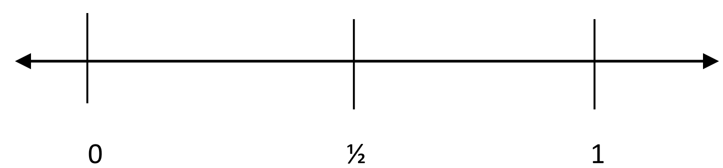
- Show that this alternating series converges absolutely

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

- Hint: recall rules about *p*-series

Estimating Alternating Series

Estimate $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}. \ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$



$S_1 =$

$S_2 =$

$S_3 =$

$S_4 =$

$$\lim_{n \to \infty} S_n = L$$

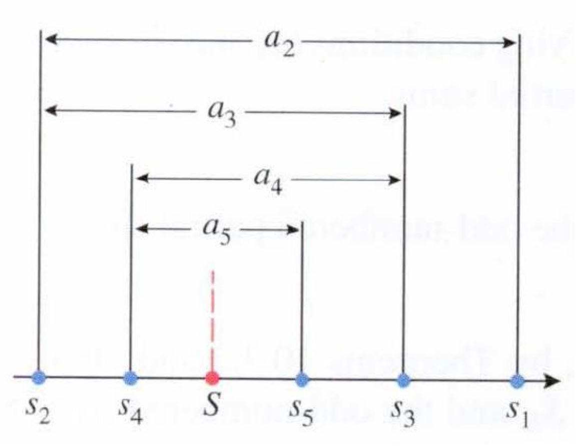
The accuracy of our estimation depends on how many terms we include in the partial sum.

<u>Error</u> is the difference between the exact value and the estimate.
Error is also called the **<u>remainder</u>**.

Error/Remainder = $$R_n = |S - S_n|$$

The maximum value of the error must be less than the next term.

$$R_n < |a_{n+1}|$$



Example: If we use the 4th partial sum to estimate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots$$

our error must be less than the 5th term.

$S_4 =$ $a_5 =$

So the actual sum must be

In other words, our sum falls between

$< S <$

You Try– *Approximating the Sum of an Alternating Series*

Approximate the sum of the following series by its first six terms.

$$\sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{1}{n!}\right) = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots$$

Solution:

The series converges by the Alternating Series Test because

$$\frac{1}{(n+1)!} \leq \frac{1}{n!} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n!} = 0.$$

Solution

cont'd

The sum of the first six terms is

$$S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144} \approx 0.63194$$

and, by the Alternating Series Remainder, you have

$$|S - S_6| = |R_6| \leq a_7 = \frac{1}{5040} \approx 0.0002.$$

So, the sum *S* lies between $0.63194 - 0.0002$ and $0.63194 + 0.0002$,

and you have $0.63174 \leq S \leq 0.63214.$