SECTION 9.9 REPRESENTING FUNCTIONS BY POWER SERIES

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \qquad |x| < 1$$

We know this is a geometric series with a =   and r =

So the sum would be

So, if – 1 < x < 1, $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$

FIND THE EQUIVALENT FUNCTION AND THE INTERVAL OF CONVERGENCE

$$\sum_{n=0}^{\infty} 5^n x^n$$

PRACTICE PROBLEMS

Find the function of x represented by the following series and state the interval of convergence.

1. $\sum_{n=0}^{\infty} 2^n x^n$ $\qquad f(x) = \frac{1}{1-2x} \qquad \frac{-1}{2} < x < \frac{1}{2}$

2. $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$ $\qquad f(x) = \frac{1}{x+2} \qquad -2 < x < 0$

3. $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-3)^n$ $\qquad f(x) = \frac{2}{x-1} \qquad 1 < x < 5$

NOW IN REVERSE, IF GIVEN A FUNCTION, REPRESENT IT AS A POWER SERIES

If you look at the geometric series as a function, it looks rather like a polynomial, but of infinite degree. Polynomials are important in mathematics for many reasons.

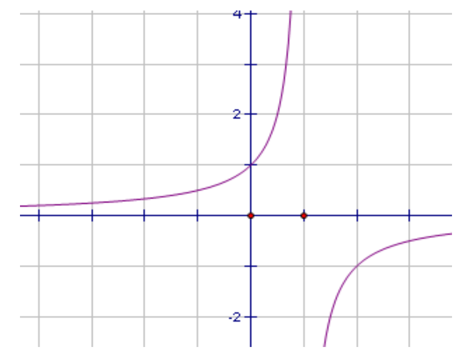
- Simplicity- they are easy to express, add, subtract, multiply, and occasionally to divide
- Closure- they stay polynomials when they are added, subtracted and multiplied
- Calculus- they are polynomials when they are differentiated or integrated.

The strategy of representing a function as a power series is useful for

- Integrating functions without elementary antiderivatives
- Solving differential equations
- Approximating functions by polynomials

Scientists do this to simplify the expressions they deal with.
Computer scientists do this to represent functions on calculators and computers

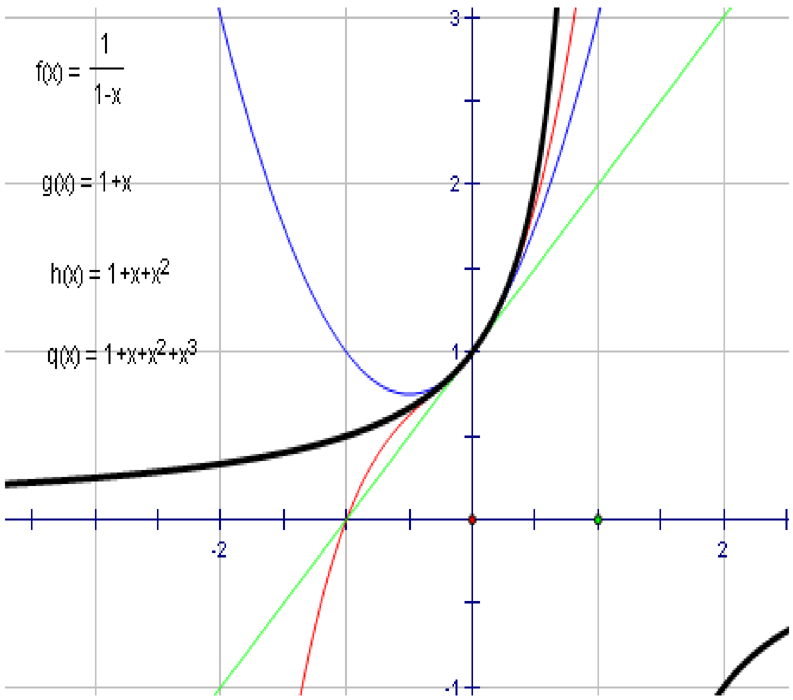
$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$



$$S_2 = 1 + x$$

$$S_3 = 1 + x + x^2$$

$$S_4 = 1 + x + x^2 + x^3$$



The function and the series behave the same on the interval of convergence. Each partial sum is an approximation of f(x). The more terms, the better the approximation.

Example 1: Find a power series representation for the function and determine the interval of convergence.

$$f(x) = \frac{1}{1+x^2}$$

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

a = 1   r = -x²

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=0}^{\infty} (-x^2)^n$$

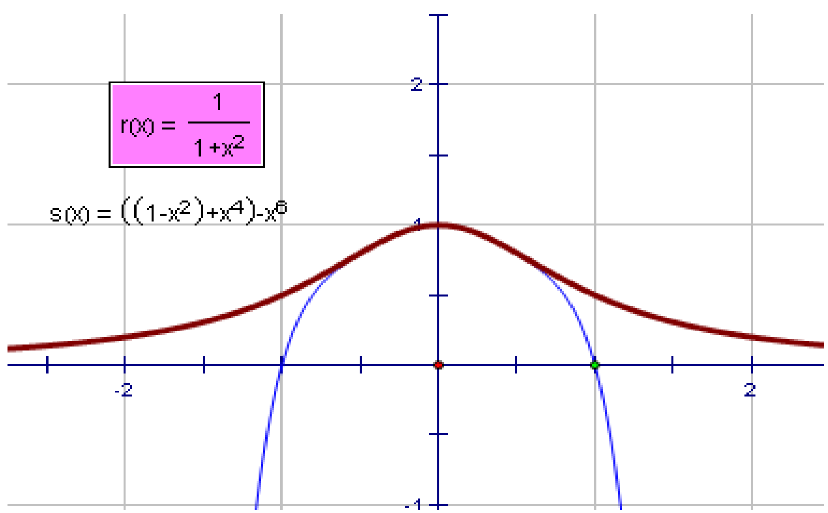
$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots$$

Example 1 – Solution continued

- Since this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$.

  - $\therefore$ the interval of convergence is $(-1, 1)$.

Geometric series diverge when r = 1, so we exclude the endpoints.



Example 2: Write $f(x)=\dfrac{1}{2+x}$ as a power series

$$f(x)=\frac{a}{1-r}$$

Remember a power series is $\sum_{n=0}^{\infty} C_n x^n$ so you must do the following:

- Separate the coefficient part from the variable
- Combine common bases into a single exponential expression
- Pull out (-1) if it's an alternating series
- Adjust exponent on (-1) to n, n – 1 , or n + 1

Example 3: Write $f(x)=\dfrac{x^3}{2+x}$ as a power series

$$f(x)=\frac{a}{1-r}$$

Another way to look at it:

LET'S ADJUST THE X EXPONENT

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} = \sum_{n=?}^{\infty} \frac{(-1)^?}{2^?} x^n$$

Do the same thing to all exponents.

$$\sum_{n=?}^{\infty} \frac{(-1)^{n-3}}{2^{n-2}} x^n$$

Do the opposite to the starting n value

$$\sum_{n=3}^{\infty} \frac{(-1)^{n-3}}{2^{n-2}} x^n$$

Readjust (-1) exponent to n, n – 1 , or n + 1

$$\sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

YOU TRY

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3^{n-1}} x^{n+2}$$

ARE THE SERIES EQUIVALENT?

$$\sum_{n=0}^{\infty} x^{n+1}$$

$$\sum_{n=1}^{\infty} x^{n}$$

Expand to find out.

$$= x + x^2 + x^3 + \cdots$$

$$= x + x^2 + x^3 + \cdots$$

They generate the same sequence, they are equivalent

$$\sum_{n=0}^{\infty} x^{2n+1}$$

$$\sum_{n=1}^{\infty} x^{2n}$$

When adjusting and simplifying, ensure the same terms are generated.

$$= x + x^3 + x^5 + \cdots$$

$$= x^2 + x^4 + x^6 \cdots$$

They do not generate the same sequence,
they are not equivalent!!!

LET'S WRITE SERIES IN EXPANDED FORM INSTEAD OF SUMMATION FORM

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_1 r^n + \cdots$$

If problem does not specify, give 4 terms

Nth term

Required!

$$f(x) = \frac{x^3}{2+x}$$

WRITE IN EXPANDED FORM

$$f(x) = \frac{1}{1+x^2}$$

$$f(x) = \frac{1}{2+x}$$

PRACTICE

$$f(x) = \frac{1}{1+3x} = 1 - 3x + 9x^2 - 27x^3 + \cdots + (-1)^n 3^n x^n + \cdots$$

a = 1 r = - 3x $(1)(-3x)^n$

$$f(x) = \frac{x}{1-2x} = x + 2x^2 + 4x^3 + 8x^4 \cdots + 2^n x^{n+1} + \cdots$$

a = x r = 2x $(x)(2x)^n$

$$f(x) = \frac{3}{1-x^2} = 3 + 3x^2 + 3x^4 + 3x^6 \cdots + 3x^{2n} + \cdots$$

a = 3 r = x^2 $(3)(x^2)^n$