SERIES

AP Exam Review

DETERMINE IF SERIES CONVERGES

- Be able to recognize geometric, harmonic, alternating, and p-series.
- Geometric series converge if $|r| < 1$.
- P-series converges if $p > 1$.
- Nth term test proves series diverges if limit of terms is not 0.
- Be able to recognize and expand a telescopic series.
- Alternating series converge if terms are decreasing and limit of terms = 0.
- Integral test can be used is function is continuous, positive and decreasing.
- Comparison tests can be used with series with positive terms.

TEST	SERIES	CONVERGENCE OR DIVERGENCE	COMMENTS
nth-term	$\sum a_n$	Diverges if $\lim_{n\to\infty} a_n \neq 0$	Inconclusive if $\lim_{n\to\infty} a_n = 0$
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	(i) Converges with sum $S = \frac{a}{1-r}$ if $\lvert r\rvert < 1$ (ii) Diverges if $\lvert r\rvert \geq 1$	Useful for the comparison tests if the nth term a_n of a series is *similar* to ar^{n-1}
p-series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	(i) Converges if $p > 1$ (ii) Diverges if $p \leq 1$	Useful for the comparison tests if the nth term a_n of a series is *similar* to $1/n^p$
Integral	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n)$	(i) Converges if $\int_1^{\infty} f(x)\,dx$ converges (ii) Diverges if $\int_1^{\infty} f(x)\,dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing, and readily integrable.
Ratio	$\sum a_n$	If $\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L$ (or ∞), the series (i) converges (absolutely) if $L<1$ (ii) diverges if $L>1$ (or ∞)	Inconclusive if $L=1$ Useful if a_n involves factorials or nth powers If $a_n>0$ for every n, the absolute value sign may be disregarded.
Root	$\sum a_n$	If $\lim_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} = L$ (or ∞), the series (i) converges (absolutely) if $L<1$ (ii) diverges if $L>1$ (or ∞)	Inconclusive if $L=1$ Useful if a_n involves nth powers If $a_n>0$ for every n, the absolute value sign may be disregarded.

Direct Comparison Test

THEOREM 9.12 DIRECT COMPARISON TEST

Let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

The Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ where $a_n, b_n > 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge

AP TIP

- Determining convergence is evaluated in the multiple choice section of the AP Exam.
- Taylor and Maclaurin series and test for convergence are tested in both the multiple choice and free response sections of the exam.
- Look for and recognize the harmonic series (which diverges) and the alternating harmonic series (which converges) as these are often used for comparison.

POWER SERIES AS GEOMETRIC SERIES

Sum of infinite geometric series:

$$S = \sum_{i=0}^{\infty} a_i r^i = \frac{a_1}{1-r}$$

[note: if |r|≥1, the infinite series does not have a sum]

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{(A geometric series.)}$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

TAYLOR SERIES

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor series centered at x = 0, is known as a MacLaurin Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

POWER SERIES

- Use ratio test to find interval of convergence of a given power series.

- Use Taylor polynomials to estimate a function.
- If the series is a convergent alternating series, the error is less than the first unused term. Be sure to mention that the series is alternating.
- For non-alternating series, use the Lagrange Error.
- C is unknown. Choose a value that will maximize the derivative, so that

$$\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|x-a\right|^{n+1}$$

Lagrange Form of the Remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Important Maclaurin Series and Their Radii of Convergence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

MEMORIZE: these Maclaurin Series, then use them to create other series.

Remember: Graphically, sine is a odd function and cosine is an even function

If the series is centered a value other than zero, you must derive the series.