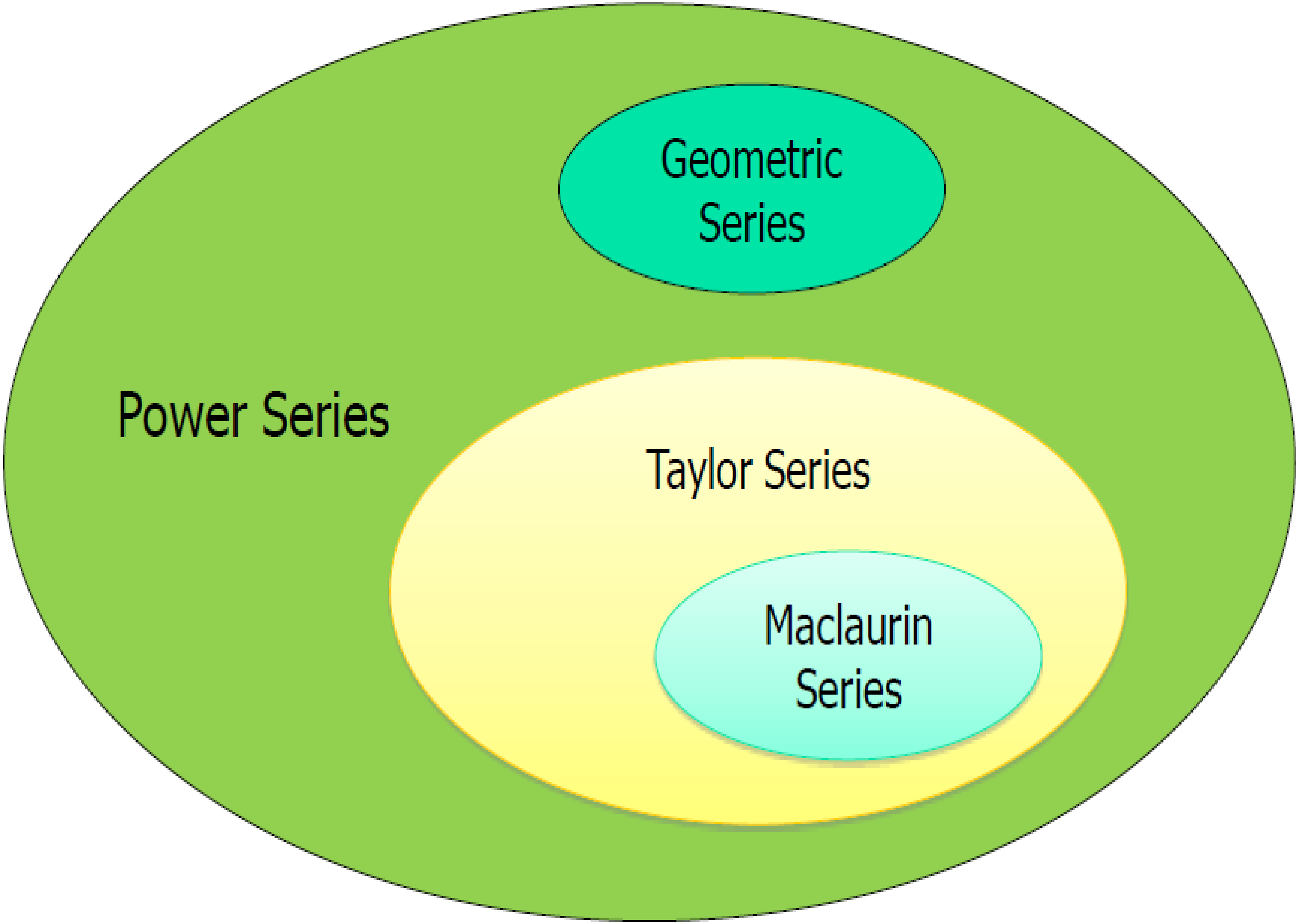
Taylor Series

Section 9.10


Geometric
Series
Power Series
Taylor Series
Maclaurin
Series

We found power series representations when functions or their integrals or derivatives were of form $\dfrac{a}{1-r}$

$$f(x) = \frac{1}{1-2x} \qquad f(x) = \tan^{-1} x$$

Today, our goal is the same but our method is different.

Taylor Series

Brook Taylor was an accomplished musician and painter. He did research in a variety of areas, but is most famous for his development of ideas regarding infinite series.

The Taylor series is named after the English mathematician Brook Taylor (1685–1731).

The Maclaurin series is named for the Scottish mathematician Colin Maclaurin (1698–1746).

Brook Taylor
1685 - 1731

Colin Maclaurin
1698-1746

Consider $f(x) = \cos x$

This is not in $\frac{a}{1-r}$ form, nor is its derivative or integral.

But we still want to find a power series for it.

$$\cos x = \sum_{n=0}^{\infty} ?$$

A series gives us a means to evaluate functions like these using polynomials.

$$f(x) = \cos x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Where is the series centered?

What happens when $x = 0$?

Everything drops out past the first term so:

$$\cos 0 = c_0 \qquad c_0 = 1$$

$$\cos x = 1 + \cdots$$

We have our first term!

$$f(x) = \cos x = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots + c_n x^n + \cdots$$

Let's take the derivative of each side

$$f'(x) = -\sin x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \cdots$$

What happens when $x = 0$?

$$-\sin 0 = c_1 \qquad c_1 = 0$$

$$\cos x = 1 + 0x + \cdots$$

$$f'(x) = -\sin x = 0 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots + nc_n x^{n-1} + \cdots$$

Let's take the 2[nd] derivative of each side

$$f''(x) = -\cos x = 2c_2 + 2\cdot 3c_3 x + 3\cdot 4c_4 x^2 + \cdots + (n-1)nc_n x^{n-2}$$

Let $x = 0$.

$$-\cos 0 = 2c_2 \qquad c_2 = -\frac{1}{2}$$

$$\cos x = 1 + 0x - \frac{1}{2}x^2 + \cdots$$

$$f''(x) = -\cos x = 2c_2 + 2\cdot 3c_3 x + 3\cdot 4c_4 x^2 + \cdots + (n-1)nc_n x^{n-2}$$

Let's take the 3[rd] derivative of each side

$$f'''(x) = \sin x = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 x + \cdots + (n-2)(n-1)nc_n x^{n-3}$$

Let x = 0

$$\sin 0 = 2\cdot 3c_3$$

$$0 = 2\cdot 3c_3$$

$$0 = c_3$$

$$\boxed{\cos x = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \cdots}$$

$$f'''(x) = \sin x = 0 + 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4 x + \cdots + (n-2)(n-1)nc_n x^{n-3}$$

Let's take the 4th derivative of each side

$$f^{iv}(x) = \cos x = 0 + 2\cdot 3\cdot 4c_4 + \cdots + (n-3)(n-2)(n-1)nc_n x^{n-4}$$

Let x = 0

$$\cos 0 = 2\cdot 3\cdot 4c_4 \qquad \frac{1}{4\cdot 3\cdot 2} = c_3$$

$$1 = 2\cdot 3\cdot 4c_3$$

$$\cos x = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4!}x^4 \cdots$$

We could continue. . .

$$\cos x = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4!}x^4 + 0x^5 - \frac{1}{6!}x^6 + \cdots$$

$$\cos x = 1 + 0x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4!}x^4 + 0x^5 - \frac{1}{6!}x^6 + \cdots$$

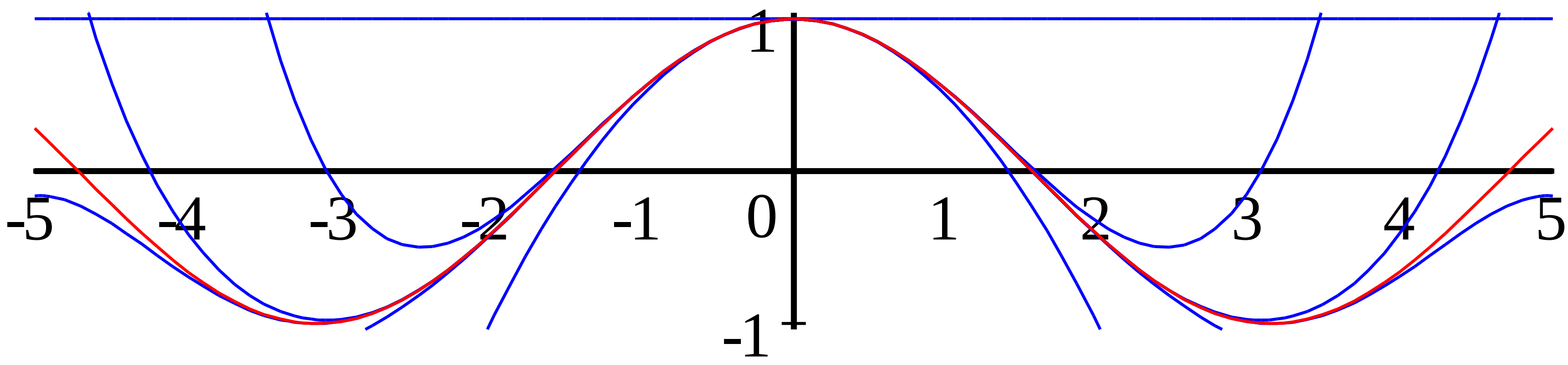
$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^? x^?}{?}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

What is the interval of convergence?

$y = \cos x$

$$P(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} \cdots$$



The more terms we add, the better our approximation.

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Let’s generalize Power Series centered at x = a

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4\cdots+c_nx^n$$

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+\cdots+nc_nx^{n-1}$$

$$f''(x)=2c_2+3\cdot 2c_3(x-a)+4\cdot 3c_4(x-a)^2+\cdots+(n-1)nc_nx^{n-2}$$

$$f'''(x)=3\cdot 2c_3+4\cdot 3\cdot 2c_4(x-a)+\cdots+(n-2)(n-1)nc_nx^{n-3}$$

$$f^n(x)=n!C_n+\cdots$$

$$f^n(a)=n!C_n$$

$$\frac{f^n(a)}{n!}=C_n$$

Taylor Series

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

A Taylor series centered at x = 0, is known as a MacLaurin Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$$

Find the Taylor Series of the function $f(x) = e^x$ and its radius of convergence centered at x = 0

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Find the Taylor Series of the function $f(x) = \ln x$ and its radius of convergence centered at x = 1

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Important Maclaurin Series and Their Radii of Convergence

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

MEMORIZE: these Maclaurin Series, then use them to create other series.

Remember: Graphically, sine is a odd function and cosine is an even function

If the series is centered a value other than zero, you must derive the series.

Find the Taylor Series of the function $f(x) = \frac{e^{2x}}{x^3}$ and its radius of convergence centered at x = 0

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Euler's Formula

An amazing use for infinite series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Substitute xi for x.

$$e^{xi} = 1 + xi + \frac{(xi)^2}{2!} + \frac{(xi)^3}{3!} + \frac{(xi)^4}{4!} + \frac{(xi)^5}{5!} + \frac{(xi)^6}{6!} + \cdots$$

$$e^{xi} = 1 + xi + \frac{x^2 i^2}{2!} + \frac{x^3 i^3}{3!} + \frac{x^4 i^4}{4!} + \frac{x^5 i^5}{5!} + \frac{x^6 i^6}{6!} + \cdots$$

$$e^{xi} = 1 + xi - \frac{x^2}{2!} - \frac{x^3 i}{3!} + \frac{x^4}{4!} + \frac{x^5 i}{5!} - \frac{x^6}{6!} + \cdots$$

Factor out the i terms.

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

→

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

This is the series for cosine.

This is the series for sine.

$$e^{xi} = \cos(x) + i\sin(x)$$

Let $x = \pi$

$$e^{i\pi} = \cos(\pi) + i\sin(\pi)$$

$$e^{i\pi} = -1 + i \cdot 0$$

$$e^{i\pi} + 1 = 0$$

This amazing identity contains the five most famous numbers in mathematics, and shows that they are interrelated.

π

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\frac{f^n(a)}{n!} = C_n$$

If you know the derivative, you can find the coefficient.

$$f^n(a) = n!C_n$$

If you know the coefficient, you can find the derivative.

Given: $f(x) = 1 - 3x^2 + 5x^4 + 8x^6 + \cdots$

Is there a relative maximum, minimum or neither at f(0)?

$$\frac{f^{n}(a)}{n!} = C_n \qquad f^{n}(a) = n!C_n$$

Given: $f(x) = 1 + 2x - 3x^2 + 6x^4 + 7x^5 + \cdots$

Is there a relative maximum, minimum or neither at f(0)?

Is f(x) increasing or decreasing at x = 0?

Is f(x) concave up or concave down at x = 0?