12.10 WS

4) $f(x) = \sin 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \mp \cdots \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

7) $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{5x} = \sum_{n=0}^{\infty} \frac{(5x)^n}{n!} = \sum_{n=0}^{\infty} \frac{5^n x^n}{n!}$

8) $xe^x = x \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

23) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\cos \pi x = \sum_{n=0}^{\infty} \frac{(-1)^n (\pi x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n} x^{2n}}{(2n)!}$

24) $e^{-x/2} = \sum_{n=0}^{\infty} \frac{\left(\frac{-x}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n n!}$

26) $\sin(4x) = \sum_{n=0}^{\infty} \frac{(-1)^n (4x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n+1} x^{2n+1}}{(2n+1)!}$

27) $x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+2}}{n!}$

28) $x \cos(2x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = x \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n)!}$

14.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3 \cdot 2x^{-4}$	$-\frac{3 \cdot 2}{16}$
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n-1}(x-2)^n}{n \cdot 2^n}$.

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{|x-2|}{2} \lim\limits_{n \to \infty} \dfrac{n}{n+1} = \dfrac{|x-2|}{2} < 1$ for convergence, so $|x-2| < 2 \Rightarrow R = 2$.

15.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$\cos x = \sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(\pi)}{k!}(x-\pi)^k = -1 + \dfrac{(x-\pi)^2}{2!} - \dfrac{(x-\pi)^4}{4!} + \dfrac{(x-\pi)^6}{6!} - \cdots = \sum\limits_{n=0}^{\infty} (-1)^{n+1} \dfrac{(x-\pi)^{2n}}{(2n)!}$.

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left[\dfrac{|x-\pi|^{2n+2}}{(2n+2)!} \cdot \dfrac{(2n)!}{|x-\pi|^{2n}} \right] = \lim\limits_{n \to \infty} \dfrac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1$ for all x, so $R = \infty$.

16.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$\sin x = \sum\limits_{k=0}^{\infty} \dfrac{f^{(k)}(\pi/2)}{k!}\left(x - \dfrac{\pi}{2}\right)^k$

$= 1 - \dfrac{(x-\pi/2)^2}{2!} + \dfrac{(x-\pi/2)^4}{4!} - \dfrac{(x-\pi/2)^6}{6!} + \cdots$

$= \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{(x-\pi/2)^{2n}}{(2n)!}$

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \left[\dfrac{|x-\pi/2|^{2n+2}}{(2n+2)!} \cdot \dfrac{(2n)!}{|x-\pi/2|^{2n}} \right] = \lim\limits_{n \to \infty} \dfrac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1$ for all x,

so $R = \infty$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$x^{-2} = 1 - 2(x-1) + 6 \cdot \dfrac{(x-1)^2}{2!} - 24 \cdot \dfrac{(x-1)^3}{3!} + 120 \cdot \dfrac{(x-1)^4}{4!} - \cdots$

$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots$

$= \sum\limits_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$.

$\lim\limits_{n \to \infty} \left| \dfrac{a_{n+1}}{a_n} \right| = \lim\limits_{n \to \infty} \dfrac{(n+2)|x-1|^{n+1}}{(n+1)|x-1|^n} = \lim\limits_{n \to \infty} \left[\dfrac{n+2}{n+1} \cdot |x-1| \right] = |x-1| < 1$ for convergence, so $R = 1$.

Supplementary College Board Type Problems

(1) $\dfrac{h^{12}(0)\, x^{12}}{12!} = \dfrac{x^{12}}{3960}$

$h^{12}(0) = \dfrac{12!}{3960} = \dfrac{479001600}{3960} = 120960 = 3(8!)$ $\boxed{B}$

(2) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$x - \sin x = \cancel{x} - x + \dfrac{x^3}{3!} - \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$

$\dfrac{x - \sin x}{x^3} = \dfrac{x^3}{x^3 3!} - \dfrac{x^5}{x^3 5!} + \dfrac{x^7}{x^3 7!} + \cdots \dfrac{(-1)^n x^{(2n+1)-3}}{(2n+1)!}$

$= \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} x^{2n-2}}{(2n+1)!}$