# MORE MOTION

# Worksheet 4. What You Need to Know About Motion Along the *x*-axis (Part 2)

1. Speed is the absolute value of ____.
2. If the velocity and acceleration have the same sign (both positive or both negative), then speed is ____.
3. If the velocity and acceleration are ____ in sign (one is positive and the other is negative), then speed is decreasing.

There are three ways to use an integral in the study of motion that are easily confused. Watch out!

4. $\int v(t)\,dt$ is an ___ ___ integral. It will give you an expression for ___ ___ at time *t*. Don't forget that you will have a ___ ___, the value of which can be determined if you know a position value at a particular time.

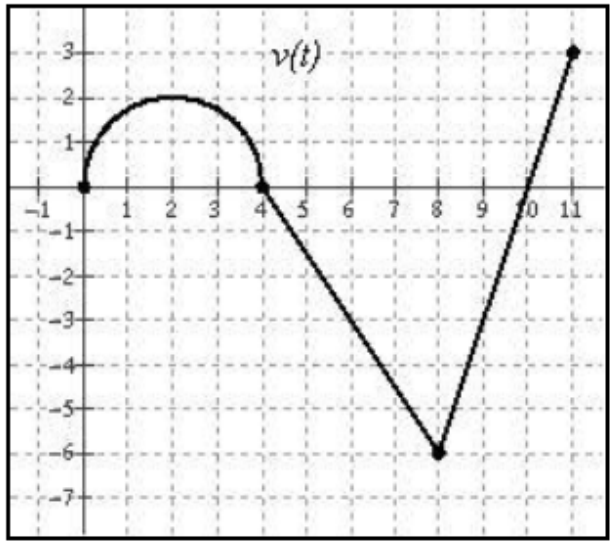
5. $\int_{t_1}^{t_2} v(t)\,dt$ is a ___ ___ integral and so the answer will be a ___ ___. The number represents the change in ___ ___ over the time interval. By the Fundamental Theorem of Calculus, since $v(t) = x'(t)$, the integral will yield $x(t_2) - x(t_1)$. This is also known as displacement. The answer can be positive or ___ ___ depending upon if the particle lands to the ___ ___ or left of its original starting position.

6. $\int_{t_1}^{t_2} |v(t)|\,dt$ is also a ___ ___ integral and so the answer will be a number. The number represents the ___ ___ traveled by the particle over the time interval. The answer should always be ___ ___.

## Worksheet 5. Sample Practice Problems for the Topic of Motion (Part 2)

### Example 1 (graphical).

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.

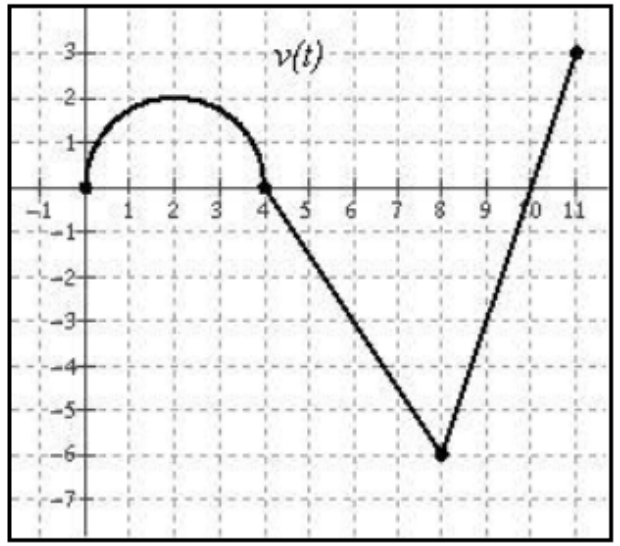



1. At what time $t$, $0 \le t \le 11$, is the speed of the particle the greatest?

1. Speed is greatest at $t = 8$ since that is when the magnitude of the velocity is the greatest.

**Example 1 (graphical).**

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.




2. At which of the times, $t = 2$, $t = 6$ or $t = 9$, is the acceleration of the particle the greatest? Explain your answer.

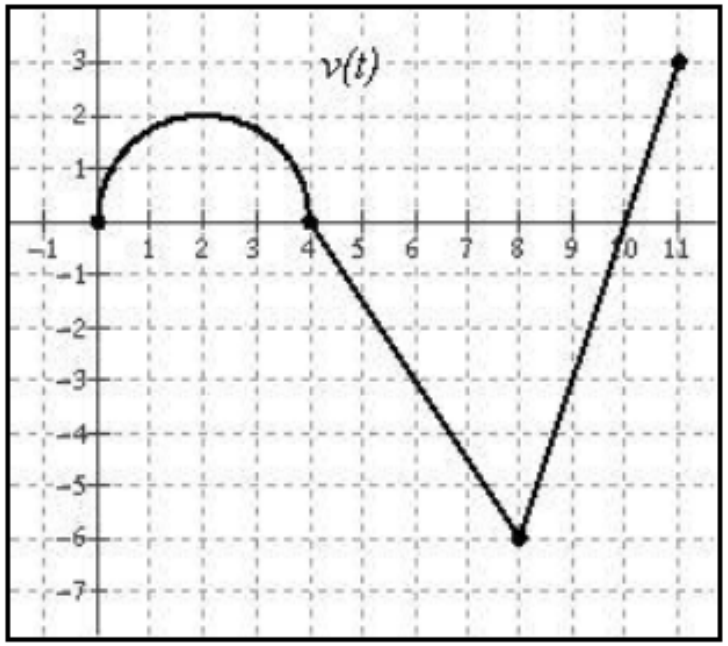
The acceleration is the derivative or slope of the velocity graph. Therefore:

$$a(2) = 0,\ a(6) = -\frac{3}{2},\ \text{and}\ a(9) = 3.$$

The acceleration of the particle is greatest at $t = 9$.

## Example 1 (graphical).

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.

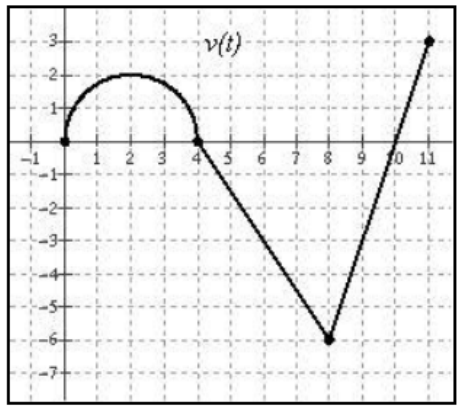



3. Over what time intervals is the particle moving to the left? Explain your answer.

The particle is moving to the left over the time interval $4 < t < 10$ because that is where the velocity is negative (velocity graph lies below the horizontal axis).

**Example 1 (graphical).**

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.




Speed is decreasing during the intervals $2 < t < 4$ and $8 < t < 10$.

4. Over what time intervals is the speed of the particle decreasing? Explain your answer.

Whenever velocity and acceleration are opposite in sign, the speed is decreasing.

velocity is positive and decreasing for $2 < t < 4$, which means $v'(t) = a(t) < 0$

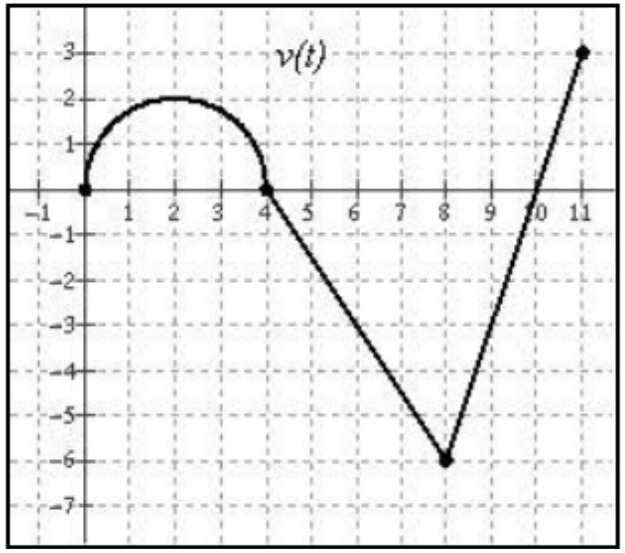
The velocity is negative and increasing for $8 < t < 10$,

which means $v'(t) = a(t) > 0$

Or, sketch and label clearly the graph of the speed function, $|v(t)|$.

**Example 1 (graphical).**

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.



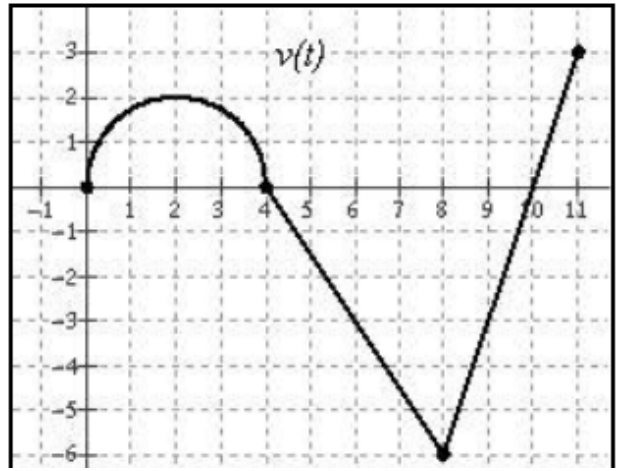

5. Find the total distance traveled by the particle over the time interval $0 \le t \le 11$.

The total distance traveled by the particle is the sum of the areas of the regions between the velocity graph and the $t$-axis. Therefore the total distance traveled is given by the expression:

$$2\pi + 18 + \frac{3}{2} = 2\pi + \frac{39}{2}$$

## Example 1 (graphical).

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.



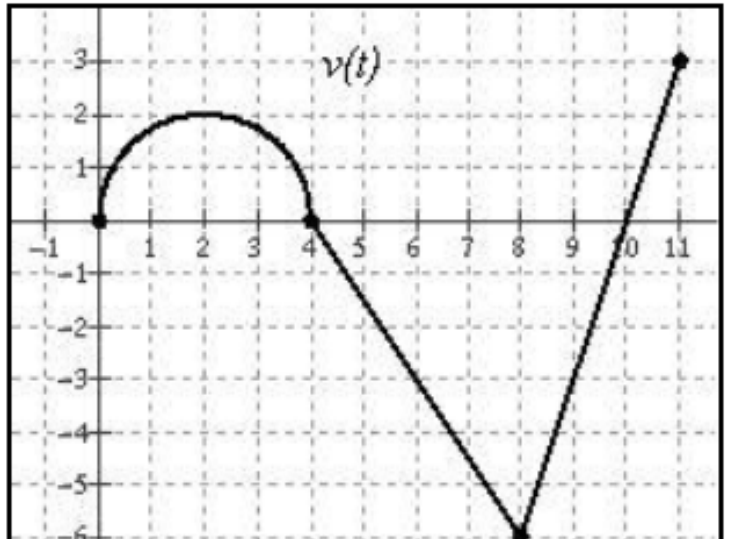

6. Find the value of $\int_0^{11} v(t)\,dt$ and explain the meaning of this integral in the context of the problem.

The definite integral shown below represents the net change in position, or the displacement, of the particle over the time interval $0 \le t \le 11$.

$$\int_0^{11} v(t)\,dt = 2\pi - 18 + \frac{3}{2} = 2\pi - \frac{33}{2}$$

## Example 1 (graphical).

The graph to the right shows the velocity, $v(t)$, of a particle moving along the $x$-axis for $0 \le t \le 11$. It consists of a semicircle and two line segments. Use the graph and your knowledge of motion to answer the following questions.




7. If the initial position of the particle is $x(0) = 2$, find the position of the particle at time $t = 11$.

Using the Fundamental Theorem of Calculus,

$$\int_0^{11} v(t)dt = x(11) - x(0)$$

$$2\pi - 18 + \frac{3}{2} = x(11) - 2$$

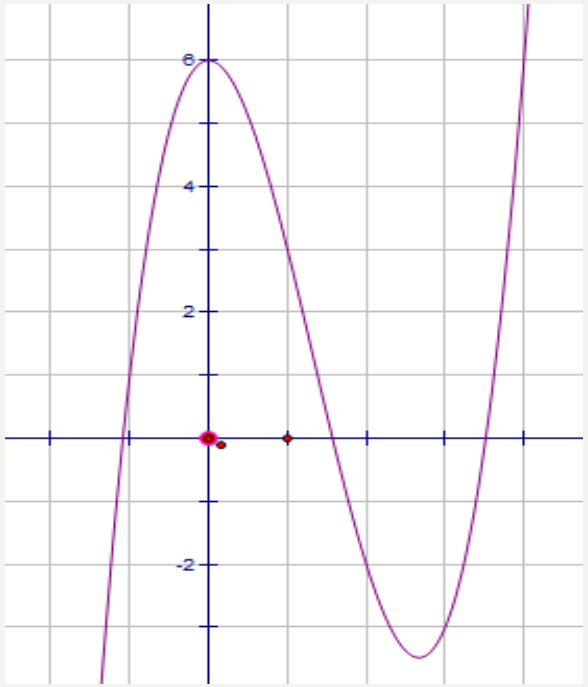
$$2\pi - 18 + \frac{3}{2} + 2 = x(11)$$

$$x(11) = 2\pi - \frac{29}{2}$$

**Example 2 (analytic/graphical/calculator active).**

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where $t$ is measured in hours. Assume the balloon is initially at ground level.

1. For what values of $t$, $0 \le t \le 4$, is the altitude of the balloon decreasing?




The altitude of the balloon is decreasing whenever $r(t) \le 0$.

using the zero feature of the calculator

$1.571993 \le t \le 3.5141369$ hours.

store these values

**Example 2 (analytic/graphical/calculator active).**

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where $t$ is measured in hours. Assume the balloon is initially at ground level.

2. Find the value of $r'(2)$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

Students can use the numerical derivative feature of their calculator

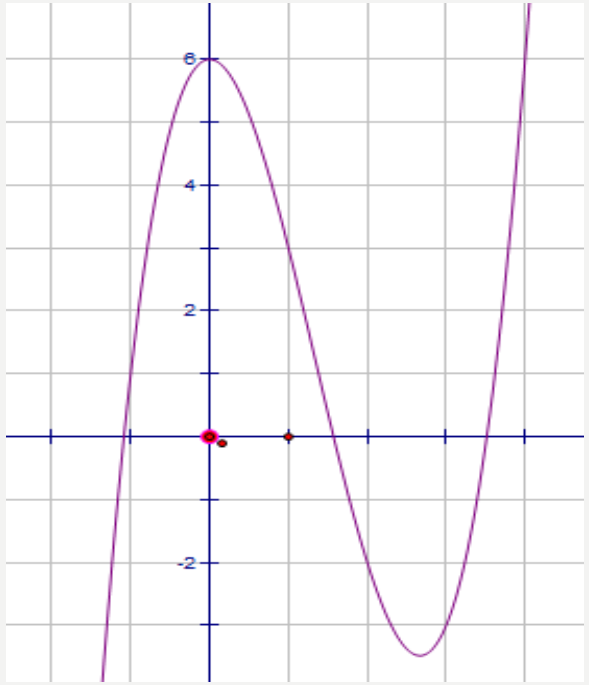
$$r'(2) = -4 \ \frac{\text{km/hr}}{\text{hr}} = -4 \frac{\text{km}}{\text{hr}^2}$$

acceleration of the balloon is $-4$ km/hr$^2$ or, equivalently, that the velocity of the balloon is decreasing at a rate of 4 km/hr$^2$.

**Example 2 (analytic/graphical/calculator active).**

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where $t$ is measured in hours. Assume the balloon is initially at ground level.

3. What is the altitude of the balloon when it is closest to the ground during the time interval $2 \le t \le 4$?




Balloon rises when velocity is positive, until t = 1.571, then decends when velocity is negative, until t = 3.514. Then rises again.
You are looking for the position at t = 3.514.

Using the Fundamental Theorem of Calculus,

$$\int_0^{3.5141369} r(t)dt = p(3.514) - p(0)$$

$$1.348 = p(3.514) - 0$$

$$1.348 = p(3.514)$$

**Example 2 (analytic/graphical/calculator active).**

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where $t$ is measured in hours. Assume the balloon is initially at ground level.

4. Find the value of $\int_0^4 r(t)\,dt$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

$$\int_0^4 r(t)\,dt = 2.667 \text{ km.}$$

The definite integral shown below represents the net change in the altitude of the balloon over the time interval $0 \le t \le 4$.

Since the balloon was initially at ground level, this definite integral also gives the altitude of the balloon at $t = 4$ hours.

**Example 2 (analytic/graphical/calculator active).**

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where $t$ is measured in hours. Assume the balloon is initially at ground level.

5. Find the value of $\int_0^4 |r(t)|\,dt$ and explain the meaning of the answer in the context of the problem. Indicate units of measure.

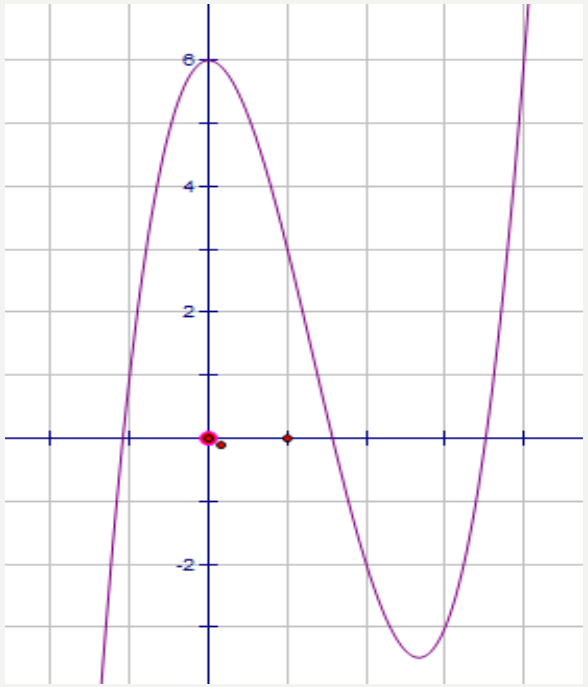
$$\int_0^4 |r(t)|\,dt = 11.529 \text{ km.}$$

The definite integral shown below represents the total vertical distance traveled by the balloon over the time interval $0 \le t \le 4$.

## Example 2 (analytic/graphical/calculator active).

The rate of change, in kilometers per hour, of the altitude of a hot air balloon is given by $r(t) = t^3 - 4t^2 + 6$ for time $0 \le t \le 4$, where $t$ is measured in hours. Assume the balloon is initially at ground level.

6. What is the maximum altitude of the balloon during the time interval $0 \le t \le 4$?




Balloon rises when velocity is positive, until t = 1.571, then decends when velocity is negative, until t = 3.514. Then rises again until 4sec. You are looking for the position at t = 1.571 or the endpoint t = 4.

Using the Fundamental Theorem of Calculus,

$$\int_0^{1.571993} r(t)dt = p(1.571) - p(0)$$

$$5.779 = p(1.571) - 0$$

$$5.779 = p(1.571)$$

$$\int_0^{4} r(t)dt = p(4) - p(0)$$

$$2.667 = p(4) - 0$$

$$2.667 = p(4)$$

Max altitude is 5.779 km.

**Example 3 (numerical).**

The table below gives values for the velocity and acceleration of a particle moving along the $x$-axis for selected values of time $t$. Both velocity and acceleration are differentiable functions of time $t$. The velocity is decreasing for all values of $t$, $0 \le t \le 10$. Use the data in the table to answer the questions that follow.

| Time, $t$ | 0 | 2 | 6 | 10 |
|---|---|---|---|---|
| Velocity, $v(t)$ | 5 | 3 | -1 | -8 |
| Acceleration, $a(t)$ | 0 | -1 | -3 | -5 |

1. Is there a time $t$ when the particle is at rest? Explain your answer.

Yes, there is a time when the particle is at rest. Since velocity is a differentiable function of time $t$, then velocity must also be continuous and the Intermediate Value Theorem can be applied. Since $v(2) = 3$ and $v(6) = -1$, then there must be a value of $t$ in the interval $2 < t < 6$ such that $v(t) = 0$.

**Example 3 (numerical).**

The table below gives values for the velocity and acceleration of a particle moving along the $x$-axis for selected values of time $t$. Both velocity and acceleration are differentiable functions of time $t$. The velocity is decreasing for all values of $t$, $0 \le t \le 10$. Use the data in the table to answer the questions that follow.

| Time, $t$ | 0 | 2 | 6 | 10 |
|---|---|---|---|---|
| Velocity, $v(t)$ | 5 | 3 | -1 | -8 |
| Acceleration, $a(t)$ | 0 | -1 | -3 | -5 |

2. At what time indicated in the table is the speed of the particle decreasing? Explain your answer.

Speed is decreasing at $t = 2$ since at that time velocity and acceleration are opposite in sign.

| Time, $t$ | 0 | 2 | 6 | 10 |
|---|---|---|---|---|
| Velocity, $v(t)$ | 5 | 3 | -1 | -8 |
| Acceleration, $a(t)$ | 0 | -1 | -3 | -5 |

3. Use a left Riemann sum to approximate $\int_0^{10} v(t)\,dt$. Show the computations you use to arrive at your answer. Explain the meaning of the definite integral in the context of the problem.

$$\int_0^{10} v(t)\,dt \approx 5\cdot 2 + 3\cdot 4 + (-1)(4) = 18$$

the definite integral represents the net change in position, or displacement, over the time interval $0 \le t \le 10$.

| Time, $t$ | 0 | 2 | 6 | 10 |
|---|---|---|---|---|
| Velocity, $v(t)$ | 5 | 3 | -1 | -8 |
| Acceleration, $a(t)$ | 0 | -1 | -3 | -5 |

4. Is the approximation found in part (3) greater than or less than the actual value of the definite integral shown below? Explain your reasoning.

$$\int_0^{10} v(t)\,dt$$

Because velocity is given to be a decreasing function on the interval $0 \le t \le 10$, the approximation in part (3) will be greater than the actual value of the integral.

| Time, $t$ | 0 | 2 | 6 | 10 |
|---|---|---|---|---|
| Velocity, $v(t)$ | 5 | 3 | -1 | -8 |
| Acceleration, $a(t)$ | 0 | -1 | -3 | -5 |

5. Approximate the value of $\int_0^{10} |v(t)|\,dt$ using a trapezoidal approximation with the three sub-intervals indicated by the values in the table. Show the computations you use to arrive at your answer. Explain the meaning of the definite integral in the context of the problem.

$$\int_0^{10} |v(t)|\,dt \approx \frac{1}{2}\cdot 2(5+3) + \frac{1}{2}\cdot 4(3+1) + \frac{1}{2}\cdot 4(1+8) = 34$$

The definite integral shown below represents the total distance traveled by the particle over the time interval $0 \le t \le 10$.

| Time, $t$ | 0 | 2 | 6 | 10 |
|---|---|---|---|---|
| Velocity, $v(t)$ | 5 | 3 | -1 | -8 |
| Acceleration, $a(t)$ | 0 | -1 | -3 | -5 |

6. Determine the value of $\int_0^{10} a(t)\,dt$. Explain the meaning of the definite integral in the context of the problem.

$$\int_0^{10} a(t)\,dt = v(10) - v(0) = -8 - 5 = -13$$

The definite integral gives the net change in the velocity of the particle over the time interval $0 \le t \le 10$.