Day 4 Review

FREE-RESPONSE QUESTION

A calculator may not be used on this question.

1. A differentiable function $f(x)$ is defined such that, for all values of x in its domain, $f(x) = 3 + \int_8^{x^3} f(\sqrt[3]{t})\,dt$.
   (a) What is the domain of $f(x)$?
   (b) For what value(s) of x is $f(x) = 3$?
   (c) Show that $f'(x) = 3x^2 f(x)$.
   (d) Solve the differential equation in (c) to find $f(x)$ in terms of x only.

$y = \frac{3}{e^8} \cdot e^{x^3} = 3e^{x^3-8}$

a) $\mathbb{R}$

b) $\int_8^{8} f(\sqrt[3]{t}) = 0$

$8 = x^3$

$x = 2$

c) $f' = 0 + f(\sqrt[3]{x^3}) \cdot 3x^2$

$f' = 3x^2 f(x)$

$\frac{dy}{dx} = 3x^2 y$

$\frac{dy}{y} = 3x^2 dx$

$\ln|y| = x^3 + C$

$y = Ae^{x^3}$ (2,3)

$3 = Ae^8$

$\frac{3}{e^8} = A$

MULTIPLE-CHOICE QUESTIONS

A calculator may not be used for the following questions.

An asterisk (*) indicates the question is for BC students.

1. The general solution to the differential equation $\frac{dy}{dx} = y^2 \sin x$ is
   (A) $y = \sqrt[3]{3\cos x + C}$
   (B) $y = -\cos x + C$
   (C) $y = \sqrt[3]{\sin x + C}$
   (D) $y = \frac{1}{\cos x + C}$

2. If $e^y \frac{dy}{dx} = 2x$ and $y(1) = 2$, then the particular solution $y(x)$ is
   (A) $y = \ln(x^2) + 2$.
   (B) $y = \ln(x^2 + e^2 - 1)$.
   (C) $y = 2e^{x^2-1}$.
   (D) $y = \ln(x^2 + e - 4)$.

1) $y^{-2} dy = \sin x\, dx$

$\frac{y^{-1}}{-1} = -\cos x + C$

$\frac{1}{y} = \cos x + C$

$\frac{1}{\cos x + C} = y$

2) $e^y dy = 2x\, dx$

$e^y = x^2 + C$

$e^2 = 1 + C$

$e^2 - 1 = C$

$e^y = x^2 + e^2 - 1$

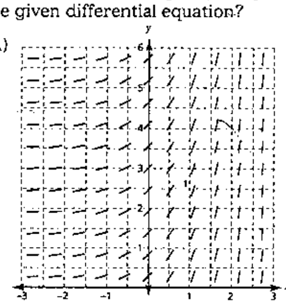
$y = \ln(x^2 + e^2 - 1)$

Questions 3-5 refer to the following information:

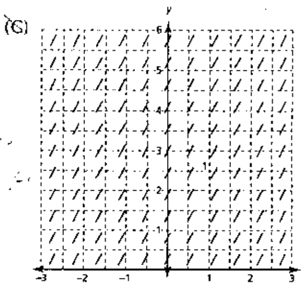
Consider the differential equation $\frac{dy}{dx} = \frac{4x}{y}$, for $y \geq 1$ only, with initial value $y(0) = 1$.

3. Which of the following is the slope field for the general solution to the given differential equation?

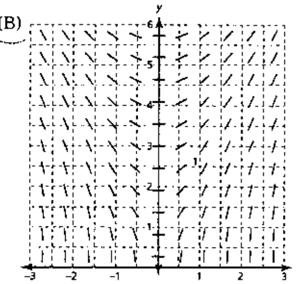
(A)



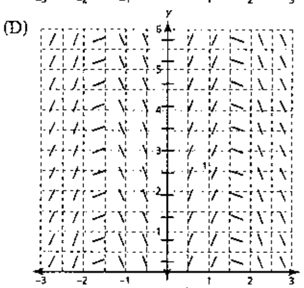
(C)



(B)



(D)



B

*4. Using Euler's Method with step size $\Delta x = 1/2$, what is the estimate for $y(1)$?
   (A) 1
   (B) 2
   (C) $\sqrt{5}$
   (D) 5/2

B

$x_1 = 0 \quad y_1 = 1$

$x_2 = .5 \quad y_2 = 1 + 0(.5) = 1$

$x_3 = 1 \quad y_3 = 1 + 2(.5) = 2$

5. The particular solution $y(x)$ is
   (A) $y = 2x$
   (B) $y = \sqrt{4x^2 - 4}$
   (C) $y = 2x^2 + 1$
   (D) $y = \sqrt{4x^2 + 1}$

D

$y\,dy = 4x\,dx$

$\frac{1}{2}y^2 = 2x^2 + C$

$\frac{1}{2} = C$

$\frac{1}{2}y^2 = 2x^2 + \frac{1}{2}$

$y^2 = 4x^2 + 1$

$y = \sqrt{4x^2 + 1}$

A calculator may be used for the following questions.

Questions 6–7 refer to the following information:

Water flows continuously from a large tank at a rate proportional to the amount of water remaining in the tank; that is, $\frac{dy}{dt} = ky$. There was initially 10,000 cubic feet of water in the tank, and at time $t = 4$ hours, 8000 cubic feet of water remained.

6. What is the value of k in the equation $\frac{dy}{dt} = ky$? $y = y_0 e^{kt}$
   (A) −0.050
   (B) −0.056
   (C) −0.169
   (D) −0.200

B

$y = 10{,}000\, e^{kt}$
$8000 = 10{,}000\, e^{4k}$

7. To the nearest cubic foot, how much water remained in the tank at time $t = 8$ hours?
   (A) 5778
   (B) 6000
   (C) 6400
   (D) 6458

C

Questions 8–10 refer to the following information:

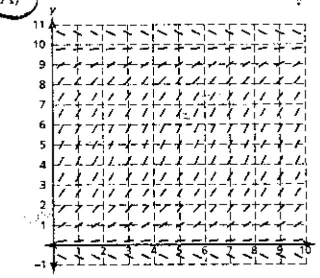
A population of rabbits in a certain habitat grows according to the differential equation $\frac{dy}{dt} = y\left(1 - \frac{1}{10}y\right)$ where t is measured in months ($t \geq 0$) and y is measured in hundreds of rabbits. There were initially 100 rabbits in this habitat; that is, $y(0) = 1$.

*8. What is the fastest growth rate, in rabbits per month, that this population exhibits?
   (A) 100
   (B) 200
   (C) 250
   (D) 500

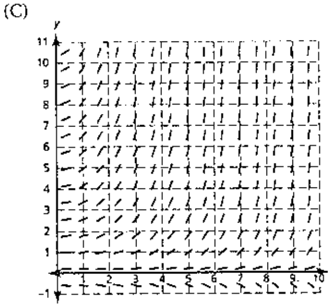
$= 10$
$y = 5$
$\frac{dy}{dt} = 5\left(1 - \frac{1}{10}(5)\right) = 2.5$ hundred/month

*9. Which of the following slope fields represents an approximate general solution to the given differential equation?

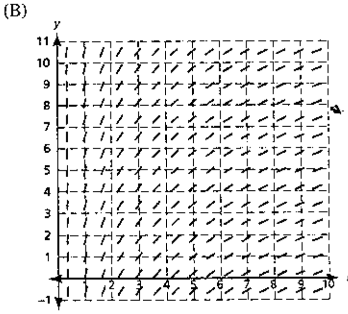
(A)



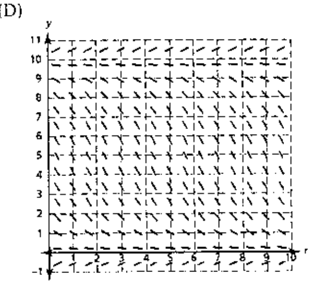
(C)



(B)



(D)



*10. Estimates of $y(t)$ can be produced using Euler's Method with step size $\Delta t = 1$. To the nearest rabbit, the estimate for $y(2)$ is
   (A) 281
   (B) 300
   (C) 344
   (D) 379

C

$1.9 + 1.9\left(1 - \frac{1.9}{10}\right)$
3.439 hundred

0	1
1	$1 + \frac{9}{10}(1) = 1.9$
2	$1.9 + 1.9\left(1 - \frac{1.9}{10}\right)$

11. Water is being pumped continuously into a tank at a rate that is inversely proportional to the amount of water in the tank; that is, $\frac{dy}{dt} = \frac{k}{y}$, where y is the number of gallons of water in the tank after t minutes ($t \geq 0$). Initially there were 5 gallons of water in the tank, and after 3 minutes there were 7 gallons. How many gallons of water were in the tank at time 18 minutes?
   (A) $\sqrt{61}$
   (B) $\sqrt{97}$
   (C) 13
   (D) 17

C

$y\,dy = k\,dt$
$\frac{1}{2}y^2 = kt + C$
$\frac{25}{2} = C$

$\frac{1}{2}y^2 = 4t + \frac{25}{2}$
$y^2 = 8t + 25$
$y = \sqrt{8t + 25}$
$y(18) = \sqrt{8 \cdot 18 + 25}$
$= \sqrt{169}$
$= 13$

$\frac{y^2}{2} = kt + \frac{25}{2}$
$\frac{7^2}{2} = k(3) + \frac{25}{2}$
$\frac{24}{2} = k3$
$4 = k$

A calculator may not be used for the following questions.

12. $f'(x) = 12x^2\sin(2x^3-16)$ and $f(2)=5$ then $f(x)=$

(A) $-4x^3\cos(2x^3-16)+5$
(B) $2\cos(2x^3-16)+3$
(C) $-2\cos(2x^3-16)+7$
(D) $-2\cos(2x^2-16)+5$

C

$u = 2x^3 - 16$
$du = 6x^2$
$2\int \sin u$
$y = -2\cos(2x^3-16)+C$
$5 = -2\cos(0)+C$
$7 = C$

13. Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}y\cos(x)$, for which the solution is $y = f(x)$. Let $f(0) = 2$.

The particular solution is
(A) $f(x) = x+2$
(B) $f(x) = 2e^{-\frac{1}{2}\sin(x)}$
(C) $f(x) = e^{\frac{1}{2}\sin(x)}$
(D) $f(x) = 2e^{\frac{1}{2}\sin(x)}$

$\frac{dy}{y}$
$\ln|y| = \frac{1}{2}\sin x + C$
$y = Ae^{\frac{1}{2}\sin x}$
$2 = A$

Questions 14 and 15 refer to the following information:

Consider the differential equation $\frac{dy}{dx} = x + 2y$, for which the solution is $g(x)$.

g'

$g'' = 1 + 2\frac{dy}{dx}$

14. Which of the following statements is true about the particular solution that contains $(0,-1)$?

(A) $g(x)$ is increasing and concave up.
(B) $g(x)$ is increasing and concave down.
(C) $g(x)$ is decreasing and concave up.
(D) $g(x)$ is decreasing and concave down.

$g' = 0 - 2$
dec

*15. Let $y(x)$ be the particular solution that contains $(0,1)$. Using Euler's Method with step size $\Delta x = \frac{1}{2}$, what is the estimate for $y\left(\frac{1}{2}\right)$?

(A) $-\frac{1}{4}$
(B) $\frac{3}{4}$
(C) $\frac{3}{2}$
(D) 2

0 1

Using this,

$x_0 = 0$ $\qquad y_0 = 1$ $\qquad \left.\frac{dy}{dx}\right|_{(0,1)} = 2$

$x_1 = \frac{1}{2}$ $\qquad y_1 = 1 + 2 \cdot \frac{1}{2} = 2$

Therefore $y\left(\frac{1}{2}\right) \approx 2$. (*Calculus for AP,* 1st ed. page 372; EU 2.3; LO 2.3F; EK 2.3F2; MPAC 2,3)

FREE-RESPONSE QUESTION

	Solution	Possible points
(a)	$\sqrt[3]{t}$ is defined for all real numbers, so x^3 can be any real number; therefore x can be any real number.	1: answer
(b)	$f(x) = 3$ when $x^3 = 8$; therefore $x = 2$ because the integral from 8 to 8 = 0.	1: answer
(c)	$f'(x) = f\left(\sqrt[3]{x^3}\right) \cdot 3x^2$, so $f'(x) = 3x^2 f(x)$.	2: $\begin{cases} 1: \text{argument of } f(x) \\ 1: \text{Chain Rule} \end{cases}$
(d)	Rewrite $f'(x) = 3x^2 f(x)$ as $\frac{dy}{dx} = 3x^2 y$. Separating variables, $\int \frac{dy}{y} = \int 3x^2\, dx$ $\ln\lvert y\rvert = x^3 + C$ $y = e^{x^3 + C} = Ce^{x^3}$. Using $f(2) = 3$ (from answer b), we get $3 = Ce^8 \Rightarrow C = 3e^{-8}$ Therefore $y = 3e^{-8}e^{x^3}$, hence $f(x) = 3e^{x^3 - 8}$.	5: $\begin{cases} 1: \text{separation of variables} \\ 2: \text{correct antiderivatives} \\ 1: \text{includes constant of integration} \\ 1: \text{solves correctly for constant} \end{cases}$

(a), (b), and (c) (*Calculus for AP* 1st ed. pages 317–325; EU 3.5,3.2,2.1,3.3; LO 3.5A,3.2C,2.1C,3.3A; EK 3.5A3,3.2C2,2.1C4,3.3A2; MPAC 2,3)
(d) (*Calculus for AP* 1st ed. pages 385–386; EU 3.5; LO 3.5A; EK 3.5A2; MPAC 2,3)