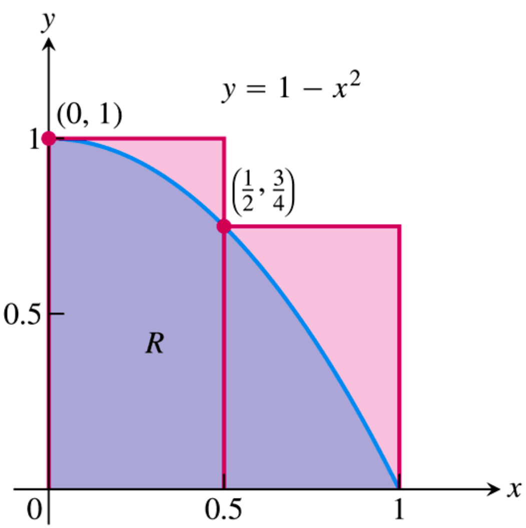
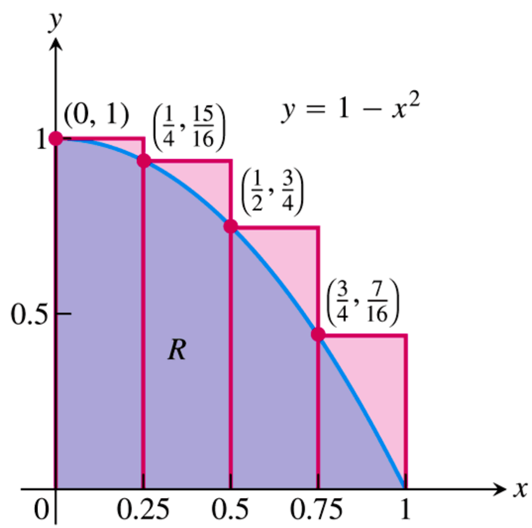
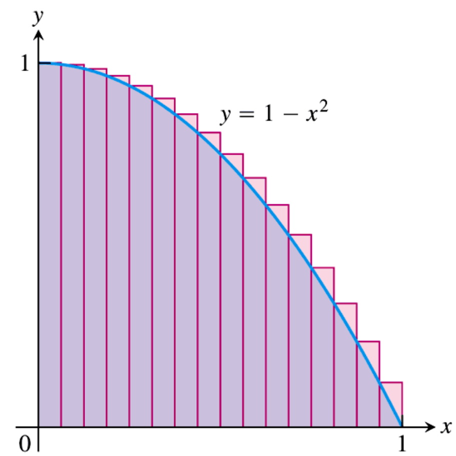
Section 5.2 The Definite Integral Using Geometry

- We have approximated the area under a curve using rectangles and trapezoids, but we would prefer to be exact.

As the number of rectangles increased, the approximation of the area under the curve approaches the exact area.







(b)

To be exact we need the number of rectangles to approach infinity

- If ***f(x)*** is a **<u>nonnegative</u>, continuous function** on the closed interval [a, b], then **the area of the region under the graph** of ***f(x)*** is given by

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

- where $\Delta x = \dfrac{b-a}{n}$
- n = number of subintervals on the interval (a,b)
- x_i = point in subinterval $x_i = a + i\Delta x$, based on right hand rectangles
- Δx is the width of our rectangles
- $f(x_i)$ is the height of our rectangle at the point x_i

This limit of the Riemann sum is also known as the definite integral of $f(x)$ on [a, b]

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)\,dx$$

This is read "the integral from a to b of *f* of *x* with respect to *x*."

Sample Exam Questions

$dx = \Delta x = \frac{b-a}{n}$ $x_k = a + k\Delta x$

Which of the following limits is equal to $\int_2^5 x^2 dx$?

(A) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{1}{n}$

(B) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{3}{n}$

(C) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{1}{n}$

(D) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{3}{n}$

Which of the following integral expressions is equal to $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1 + \frac{3k}{n}} \cdot \frac{1}{n} \right)$?

(A) $\int_0^1 \sqrt{1 + 3x}\, dx$

(B) $\int_0^3 \sqrt{1 + x}\, dx$

(C) $\int_1^4 \sqrt{x}\, dx$

(D) $\frac{1}{3} \int_0^3 \sqrt{x}\, dx$

Notation for the definite integral

upper limit of integration

Integration
Symbol
(integral)

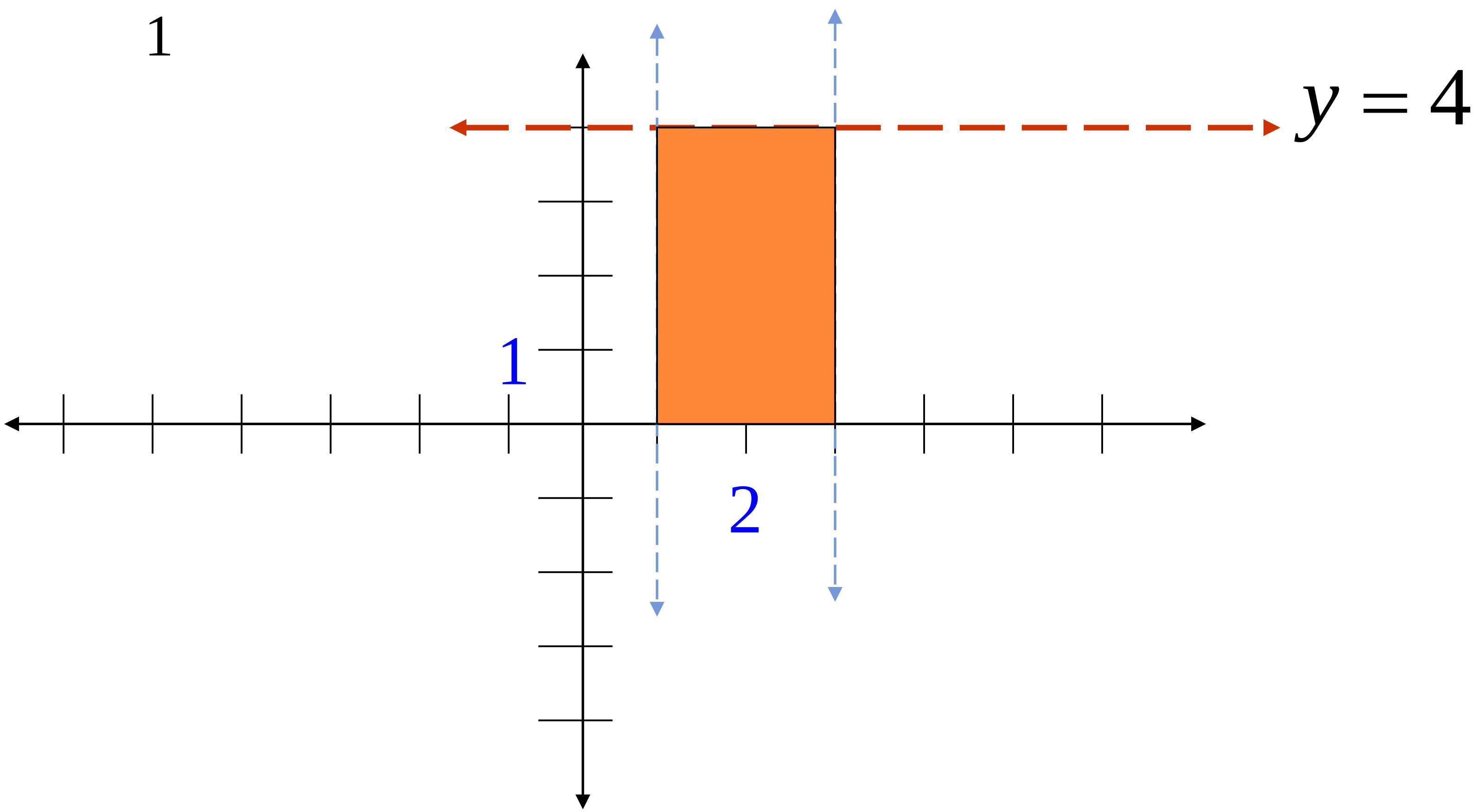
$$\int_a^b \underbrace{f\left(x\right)}_{\text{integrand}} dx$$

variable of integration

lower limit of integration

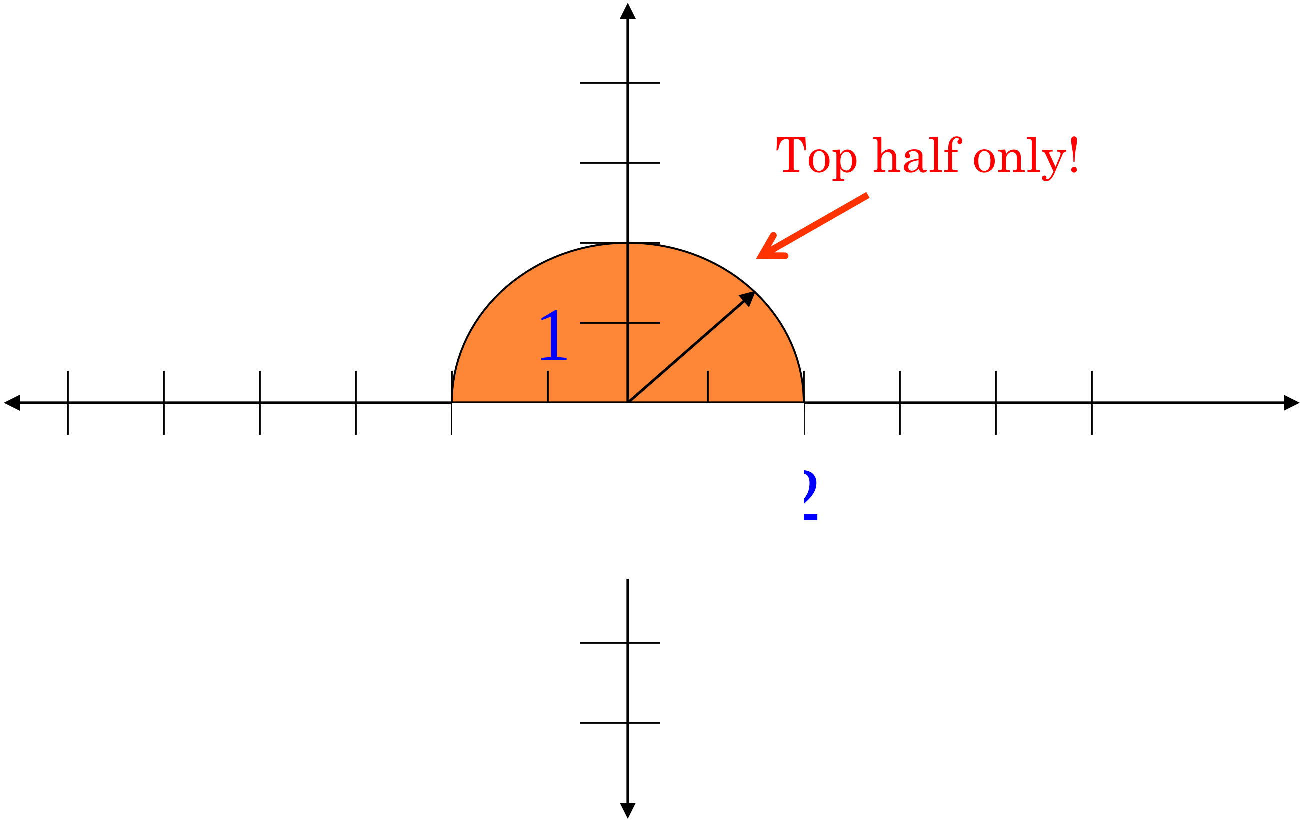
Evaluate the following definite integrals using geometric area formulas.

$$\int_{1}^{3} 4dx$$

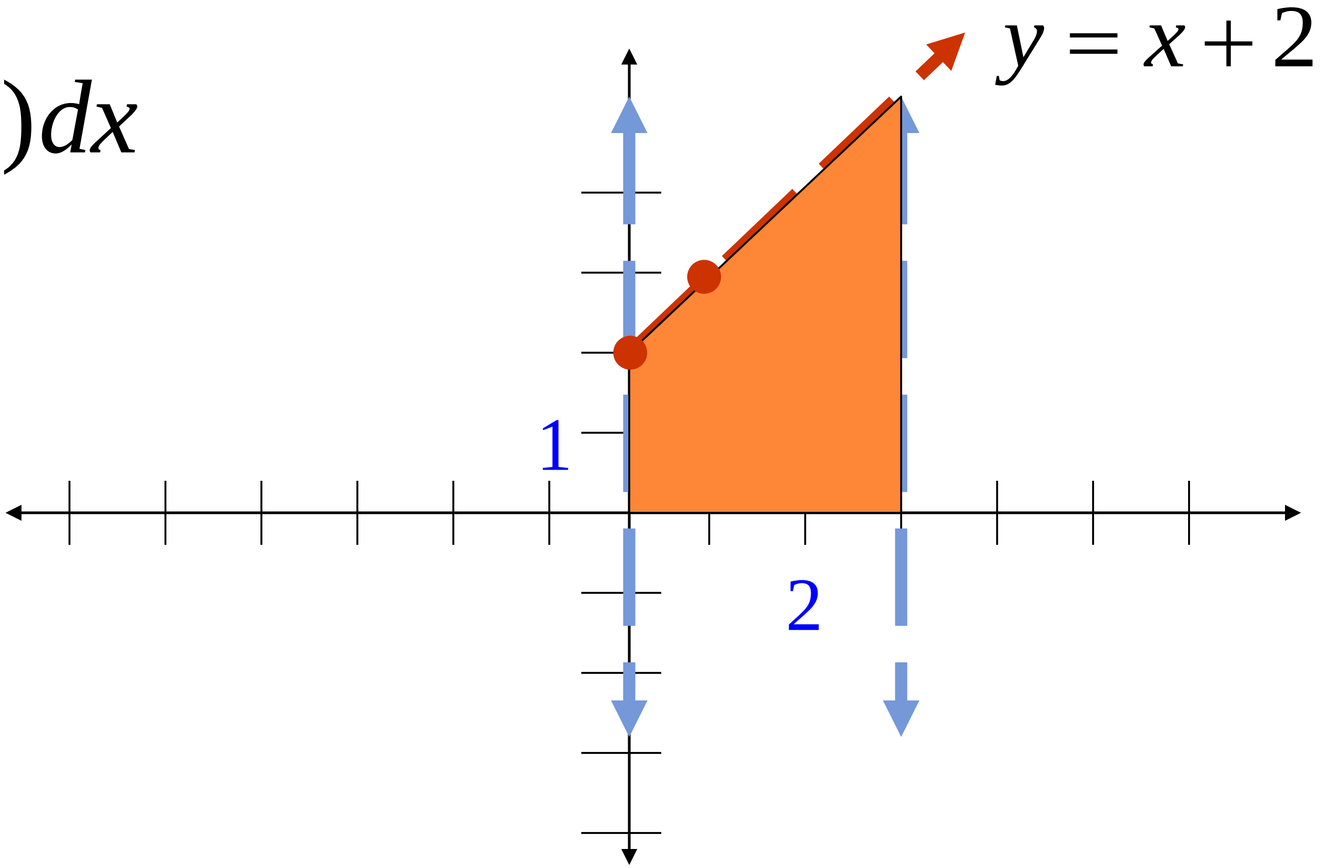


$$\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$y=\sqrt{4-x^2} \Rightarrow x^2+y^2=4$$

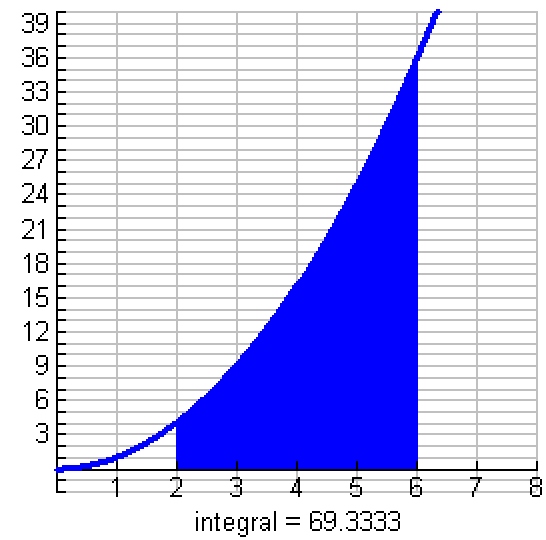


$$\int_0^3 (x+2)\,dx$$



We cannot use a formula to find the area under a curve, so we will use the calculator.

fnInt is option 9 under MATH key



fnInt(function, variable of integration, lower bound, upper bound)

$$\int_2^6 x^2\,dx = fnInt\left(x^2, x, 2, 6\right) = 69\frac{1}{3}$$

Evaluate the following integrals using your calculator

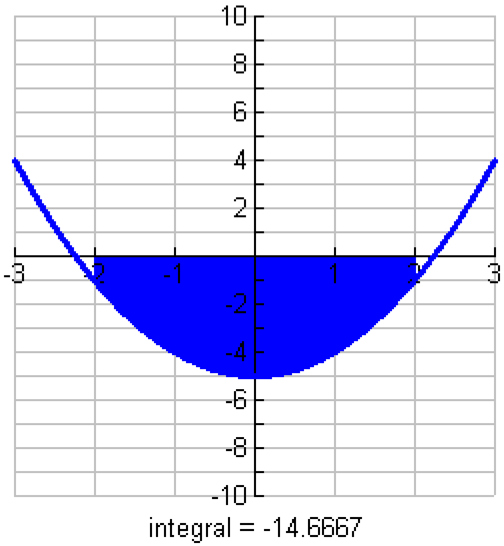
$$\int_0^5 3x^2 + 2x \, dx \qquad = 150$$

$$\int_{-2}^8 4x^2 + 3x \, dx \qquad = 783\frac{1}{3}$$

$$\int_{-4}^4 6x^2 \, dx \qquad = 256$$

Evaluate the following integral using your calculator

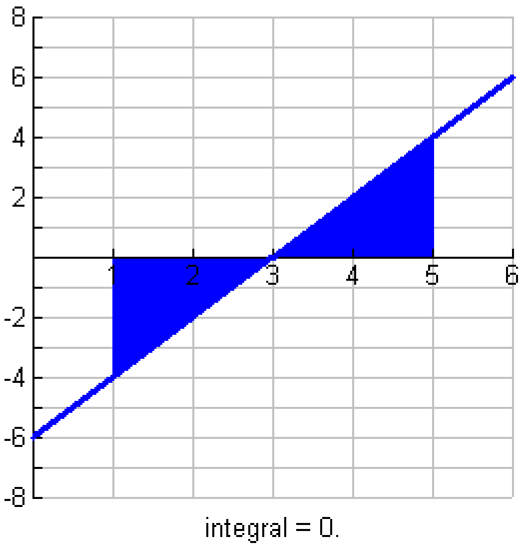
$$\int_{-2}^{2} x^2 - 5\, dx = -14\frac{2}{3}$$



Integrals treat areas <u>beneath</u> the x-axis as <u>*negative*</u>

Evaluate the following integral using your calculator

$$\int_1^5 2x - 6\, dx = 0$$



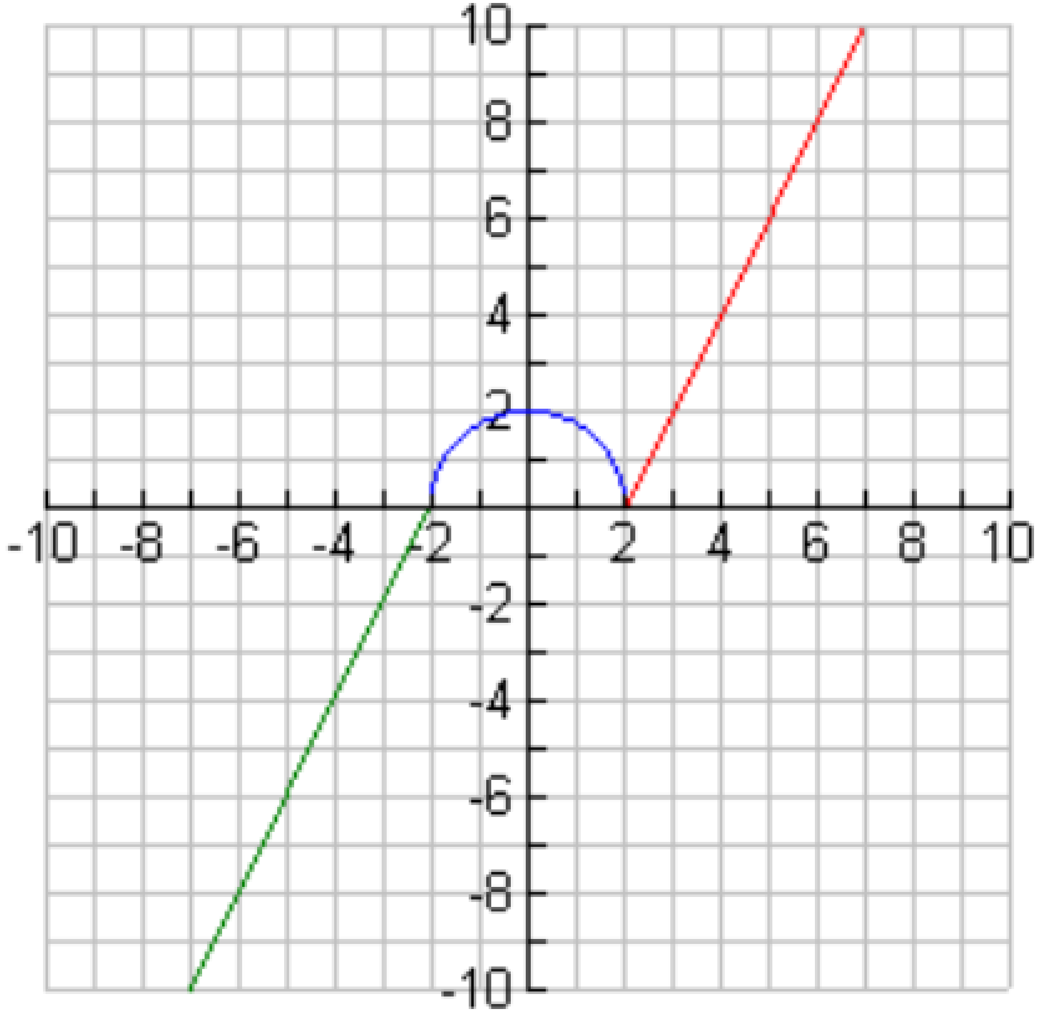
Areas above and below the x-axis can cancel each other out.

For $f(x)$ shown, Find

$$\int_{-2}^{2} f(x)dx$$

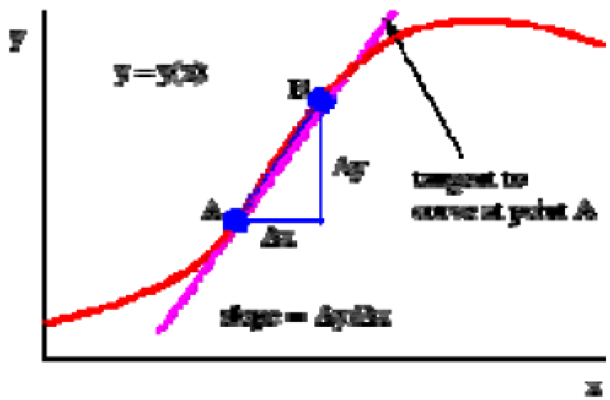
$$\int_{-6}^{-2} f(x)dx$$

$$\int_{-6}^{6} f(x)dx$$

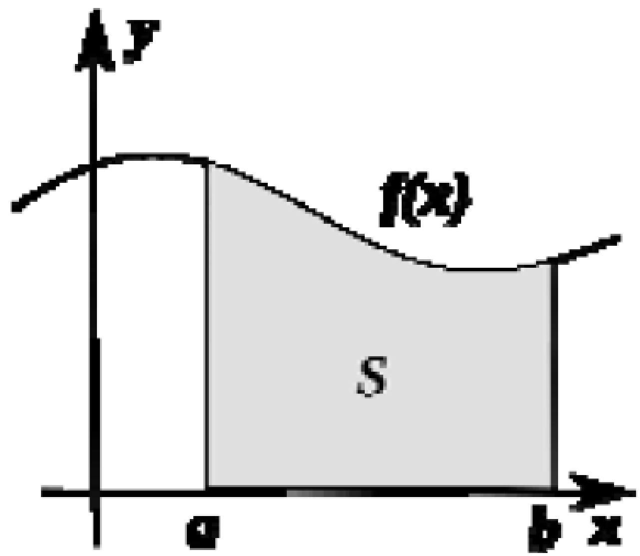


Derivatives vs Integrals

Derivatives are about rates of change and slopes of graphs.



Integrals are about accumulation and areas of graphs.



Units

The units of a derivative equal the quotient of the dependent and independent axes.

$$\frac{dy}{dx} = \frac{y \text{ units}}{x \text{ units}} \quad \text{slopes are ratios}$$

The units of an integral or Riemann sum equal the product of the independent and dependent axes.

$$\text{Area} = \Delta x \cdot f(x) = (x \text{ units})(f \text{ units})$$

$$\text{Areas are products}$$

Part 1 – Estimating Distance Traveled

A car is traveling so that its speed never decreases during a 10-second interval. The velocity at various points in time is listed in the table below.

Time (seconds)	0	2	4	6	8	10
Velocity (ft/sec)	30	36	40	48	54	60

If we want to know how far the car moved, we "integrate" the velocity function.

$$\int_{0}^{10} v(t)\,dt = \text{ displacement (how far the car moved)}$$

In this activity, travel is in only one direction, so speed and velocity are the same and total distance and displacement are the same.

Riemann Sums are used to estimate integrals.

Derivatives are like a speedometers, which show **the instantaneous rate of change**

Integrals are like odometers, which show total miles **accumulated** over some interval

Other applications

$$\int_a^b \text{rate} \ = \text{accumulation}$$

$$\int \text{rate of oil leaking } dt = \ \text{total oil leaked}$$

$$\int \text{rate of population growth } dt = \ \text{total population}$$

$$\int \text{rate a medication is absorbed into bloodstream } dt = \ \text{total medication}$$

$$\int \text{rate energy is released from a chemical reaction } dt = \ \text{total energy}$$