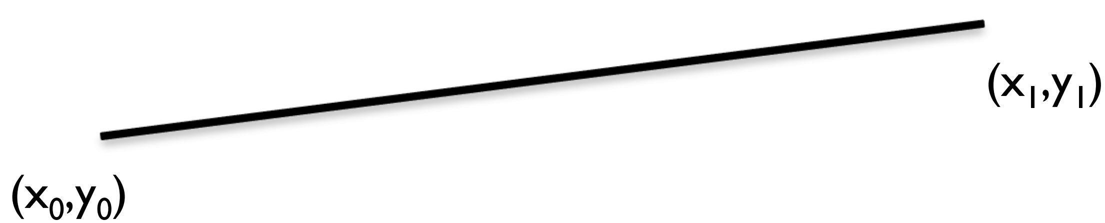
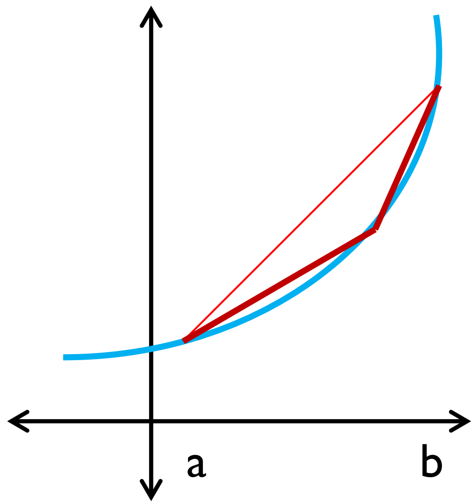
Section 6.4 Arc length and Surface Area of Revolution

Start with something easy

The length of the line segment joining points (x_0,y_0) and (x_1,y_1) is

$$\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2}$$





We could approximate the length of a curve, f(x), from a to b using a segment connecting the endpoints.

If we divide the curve into two intervals and add up the length of two segments, we get a better approximation.

The more segments, the better the approximation.

For a demonstration, let's visit the web.

Arc length, L = $\int_a^b \sqrt{(x - x)^2 + (y - y)^2}$

$$L = \int_a^b \sqrt{(dx)^2 + (dy)^2}$$

This is not quite in the form that we prefer, since we need a *dx* or *dy* outside the square root. So, if we multiply by $\frac{(dx)^2}{(dx)^2}$

$$L = \int_a^b \sqrt{\left((\frac{dx}{dx})^2 + (\frac{dy}{dx})^2 \right) * (dx)^2} = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

$\frac{dy}{dx}$ is the slope between two points. And as our points get closer and closer together, we get an instantaneous slope, or the first derivative!

The Formula:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Example Problem

- Compute the arc length of the graph of $f(x) = x^{3/2}$ over [0,1].

$$L = \int_0^1 \sqrt{1 + \left[f'(x) \right]^2}\, dx \qquad f'(x) = \frac{3x^{1/2}}{2}$$

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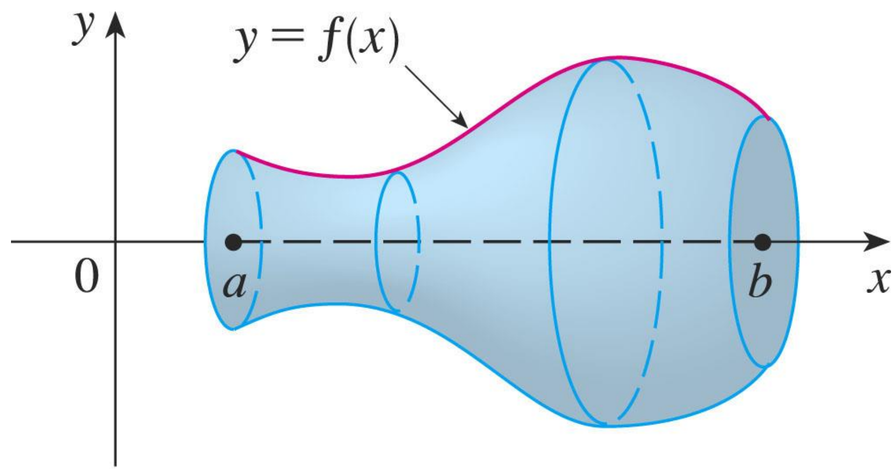
$$L \approx 1.44$$

What are we doing next?

- Instead of calculating the volume of the rotated surface, we are now going to calculate the surface area of the solid of revolution

The Lateral Surface Area of a Cylinder (label on a can) = circumference * height = 2πrh

When you rotate a function , the "height" is the length of the curve.



(a) Surface of revolution

$$SA = \int_a^b 2\pi r \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

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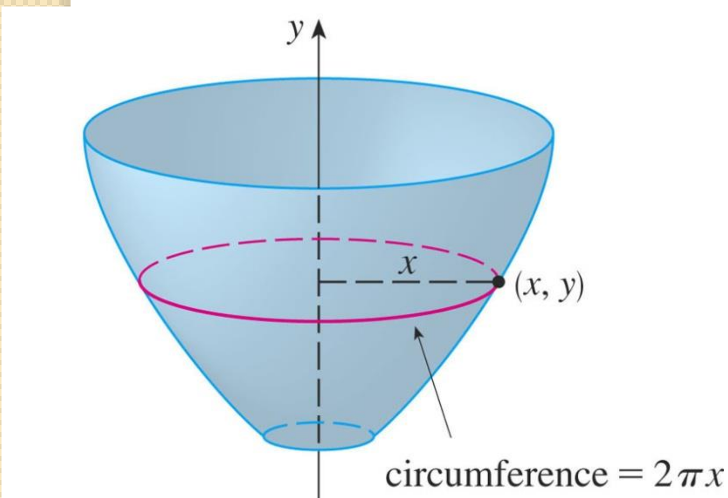
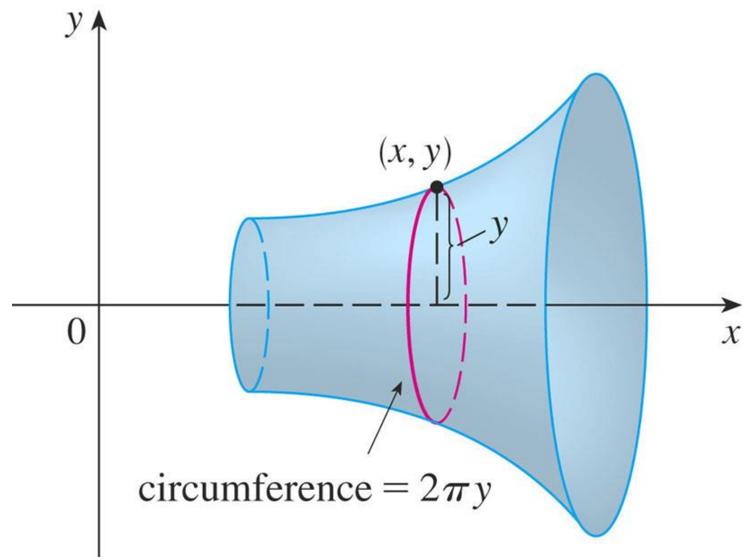


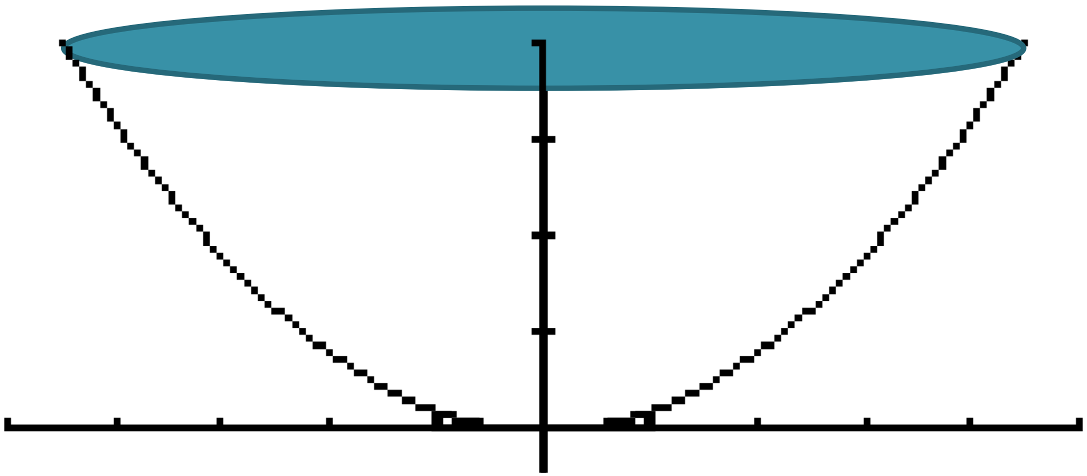


When rotating around a vertical line, r = x.

When rotating around a horizontal line, r = y = f(x).

Surface Area (examples)

Calculate the surface area when the curve f(x) = x^2 on [0 , 4] is rotated about the y axis.



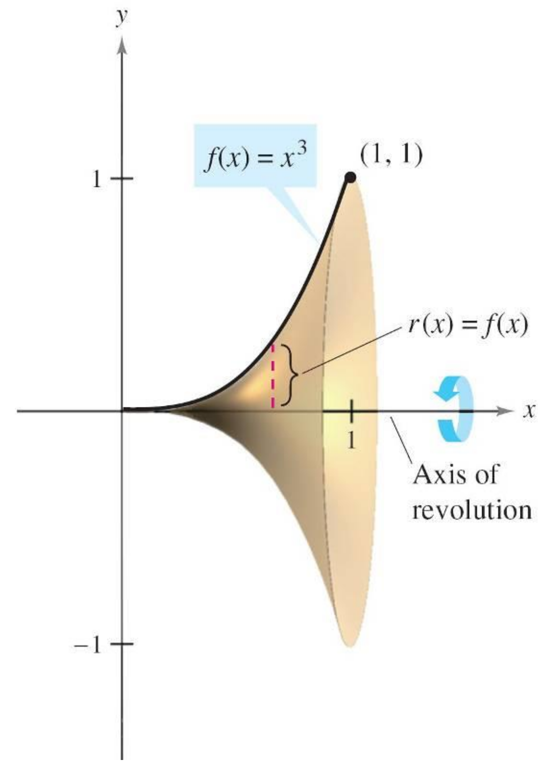
$$\textbf{SA} = \int_a^b 2\pi r\sqrt{1+[f'(x)]^2}\,dx = \int_0^4 2\pi x\sqrt{1+(2x)^2}\,dx \quad = 273.867$$

whose value can be done with a u- substitution or a fnInt on a calculator.

Find the area of the surface formed by revolving the graph of $f(x) = x^3$ on the interval [0, 1] about the x-axis.

$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx =$$

$$= 2\pi\int_0^1 x^3\sqrt{1 + (3x^2)^2}\,dx \quad \approx 3.563.$$



Can you solve this one by hand?

$$2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx$$

$$= \frac{2\pi}{36}\int_0^1 (36x^3)(1 + 9x^4)^{1/2}\,dx = \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]_0^1 = \frac{\pi}{27}(10^{3/2} - 1)$$

Practice: p. 444 #5, 7, 9, 35-43 odd

Quiz on Friday