Section 7.7 Indeterminate Forms

Review: $\lim_{x \to 2} \frac{x^2 + 6x - 16}{x - 2}$

$$= \frac{2^2 + 6 \cdot 2 - 16}{2 - 2} = \frac{0}{0}$$

We end up with an indeterminate form

Note <u>why</u> this is indeterminate

$$\frac{0}{0} = n \quad \Rightarrow 0 \cdot n = 0 \quad n = ?$$

$$\lim_{x \to 2} \frac{x^2 + 6x - 16}{x - 2}$$

We can evaluate it by factoring and canceling:

$$= \lim_{x \to 2} \frac{\cancel{(x - 2)}(x + 8)}{\cancel{x - 2}}$$

$$= \lim_{x \to 2} x + 8 = 2 + 8 = 10$$

Review: $$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}=\frac{3(\infty)^2-1}{2(\infty)^2+1}=\frac{\infty}{\infty}$$

We end up with another indeterminate form

$$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}$$

Divide numerator and denominator by x^2
(highest power in the denominatior)

$$=\lim_{x\to\infty}\frac{3-(1/x^2)}{2+(1/x^2)}=\frac{3-0}{2+0}=\frac{3}{2}$$

Or use L'Hôpital's Rule

If we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞,

then $$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to 2}\frac{x^2+6x-16}{x-2}=\lim_{x\to 2}\frac{2x+6}{1}=\frac{2\cdot 2+6}{1}=10$$

$$\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1}=\lim_{x\to\infty}\frac{6x}{4x}=\frac{6}{4}=\frac{3}{2}$$

Try: $\lim_{x \to 1} \frac{\ln x}{x-1} = \frac{0}{0}$

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

Try: $$\lim_{x\to\infty}\frac{e^x}{x^2} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{e^x}{2x} = \frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\frac{e^x}{2} = \frac{\infty}{2} = \infty$$

Remember that L'Hôpital's Rule can be applied only to quotients leading to the indeterminate forms $\frac{\infty}{\infty}$ or $\frac{0}{0}$

There are similar forms that you should recognize as "determinate."

$$\frac{\#}{\infty} \to 0$$

$$\frac{\#}{0} \to \pm\infty$$

$$0^{\infty} \to 0$$

$$\frac{\infty}{\#} \to \pm\infty$$

$$\infty + \infty \to \infty$$

$$0^{-\infty} \to \pm\infty$$

$$-\infty - \infty \to -\infty$$

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad \infty - \infty \qquad 1^{\infty} \qquad 0^{0} \qquad \infty^{0} \qquad 0 \cdot \infty$$

have been identified as *indeterminate*.

The first two can be evaluated using L'Hôpital directly.

$\infty - \infty \; and \; 0 \cdot \infty$ need to be rewritten as a fraction in order to use L'Hôpital.

We will deal with the indeterminate powers tomorrow.

$$\lim_{x \to 0+} x \ln x$$

$$\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x)$$

Try:

$$\lim_{x \to \pi^-} (x - \pi) \cot x = 1 \qquad \lim_{x \to 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = 1/2$$

$$\lim_{x\to\pi-}(x-\pi)\cot x = 0 \cdot -\infty$$

$\frac{\cos\pi}{\sin\pi} = \frac{-1}{0} = -\infty$ because we are approaching from the left, so sin π is positive

$$= \lim_{x\to\pi-}\frac{x-\pi}{\tan x} = \frac{0}{0}$$

$$\overset{H}{=} \lim_{x\to\pi-}\frac{1}{\sec^2 x} = \lim_{x\to\pi-}\cos^2 x = (-1)^2 = 1$$

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$ ← This is indeterminate form $\infty-\infty$

Rewrite as a ratio!

If we find a common denominator and subtract, we get:

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)$$ ← Now it is in the form $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\frac{x-1}{x}+\ln x}\right)$$ ← L'Hôpital's rule applied once.

$$\lim_{x\to 1}\left(\frac{x-1}{x-1+x\ln x}\right)$$ ← Fractions cleared. Still $\frac{0}{0}$

$$\lim_{x\to 1}\left(\frac{1}{1+1+\ln x}\right)$$ ← L'Hôpital again.

Answer: $\frac{1}{2}$

Practice

p. 511 #7-39 every other odd,
44, 46, 58, 60, 79

Even Answers: #44 is 0
#46 is 1, #58 is -1/8, and
#60 is $-\infty$

Section 7.7 Indeterminate Forms continued

Indeterminate Powers

Indeterminate Powers

$$\lim_{x \to a} [f(x)]^{g(x)} \to 0^0, \quad \infty^0, \quad 1^\infty$$

Idea :

these three cases can be treated by taking the natural logarithm:

$$y = [f(x)]^{g(x)} \to \ln y = g(x) \ln[f(x)]$$

Example $\lim_{x \to 0+} x^x$

$\lim_{x \to 0^+} x^x = 0^0$ Direct Substitution

$y = \lim_{x \to 0^+} x^x$ Take natural log of both sides

$lny = ln \lim_{x \to 0^+} x^x$ Move log inside the limit

$lny = \lim_{x \to 0^+} ln\, x^x$ Use log rule

$lny = \lim_{x \to 0^+} xln\, x = 0 \cdot (-\infty)$

$lny = \lim_{x \to 0^+} \frac{lnx}{\frac{1}{x}} = \frac{-\infty}{\infty}$ Now use L'H.

$lny = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} =$ Simplify

$lny = \lim_{x \to 0^+} -x = 0$

$lny = 0$ This is not the final answer.

$e^{lny} = e^0$

$y = 1$

$y = \lim_{x \to 0^+} x^x = 1$

Try: $\lim_{x \to 0+} (-\ln x)^x = 1$ $\lim_{x \to \infty} \left(x^2 + e^x\right)^{5/x} = e^5$

- p. 511 #47 – 56 all, 81, 82