Section 7.8 Improper Integrals

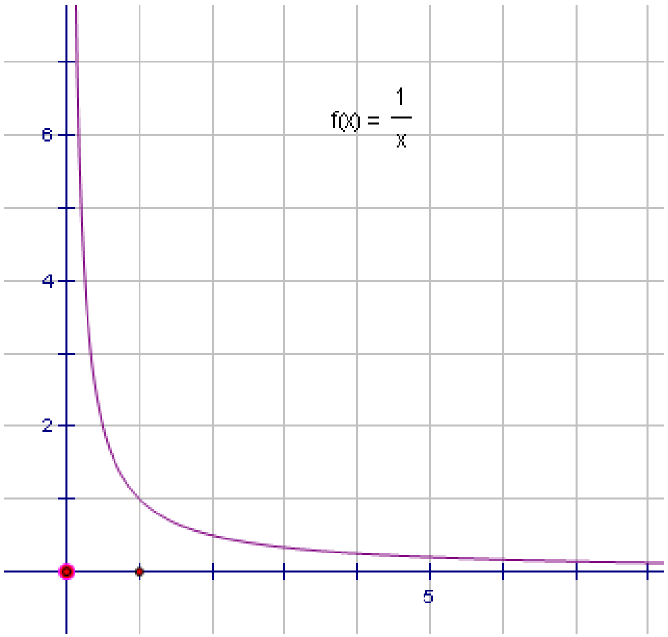
- Improper integrals have limits of integration that are infinite or the function has a finite number of infinite discontinuities on a certain interval.

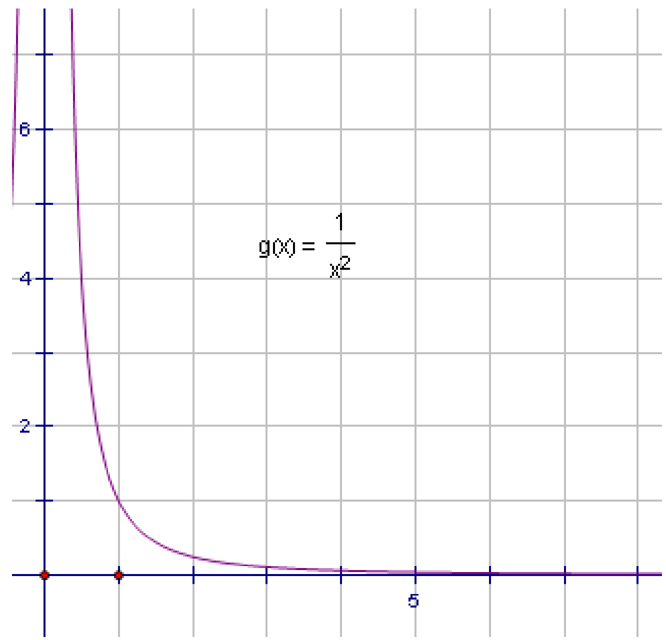
$$\int_{1}^{\infty} \frac{dx}{x} \qquad \int_{-\infty}^{\infty} \frac{dx}{x^2+1} \qquad \int_{1}^{5} \frac{dx}{\sqrt{x-1}} \qquad \int_{-2}^{2} \frac{dx}{(x+1)^2}$$

- These integrals may DIVERGE (continue to grow and grow) or CONVERGE to a particular value (you will get a numerical answer)

You cannot just look at the graph and tell.

$$\int_1^\infty \frac{1}{x}dx \qquad\qquad \int_1^\infty \frac{1}{x^2}dx$$





One of these converges, and the other diverges.

Example 1a: $\int_1^\infty \frac{1}{x}dx$

To integrate we cannot just plug in infinity, we need to *approach* infinity. We need limits!

$$\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b$$

Evaluate the integral and substitute the limits of integration first.

$$= \lim_{b\to\infty}(\ln b - \ln 1)$$

Then evaluate the limit.

$$= \infty - 0 = \infty$$

This integral diverges.

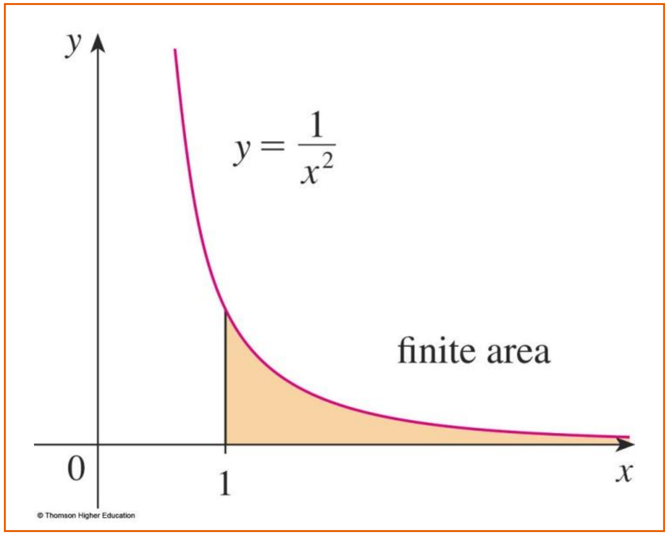
Example 1b: $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$

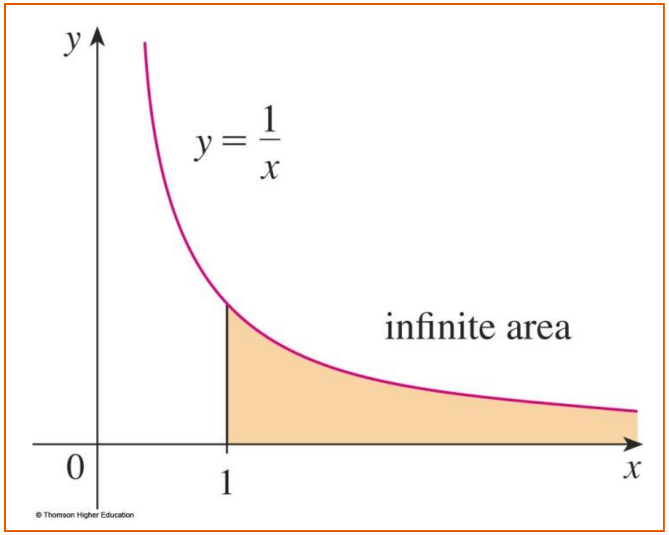
$$\lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1}\right)$$

$$=\left(-\frac{1}{\infty}+\frac{1}{1}\right) = -0+1=1$$

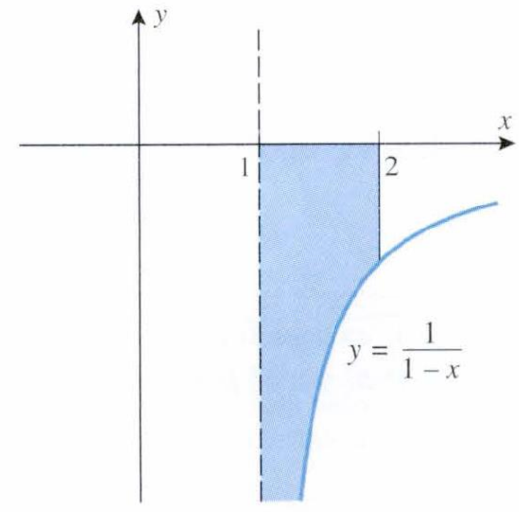
This integral converges to 1.

- Note that both $1/x^2$ and $1/x$ approach 0 as $x \to \infty$, but $1/x^2$ approaches faster than $1/x$.
  - The values of $1/x$ don't decrease fast enough for its integral to have a finite value.





Example 2 $\int_1^2 \frac{dx}{1-x}$



This function is discontinuous at x = 1,
so we must approach 1 from the right.

$$\int_1^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \int_k^2 \frac{dx}{1-x} = \lim_{k\to 1^+} \left[-\ln|1-x|\right]_k^2$$

$$= \lim_{k\to 1^+} \left(-\ln|-1| + \ln|1-k|\right)$$

$$= -0 + -\infty = -\infty$$

Integral Diverges.

Try: $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx = \lim_{b\to 4-} \int_0^b (4-x)^{-1/2}\,dx$

Discontinuous at x = 4.
Need to approach 4 from the left

$0 \to 4$

$$= \lim_{b\to 4-} \left[\frac{-(4-x)^{1/2}}{1/2} \right]_0^b$$

$$= \lim_{b\to 4-} \left[-2(4-b)^{1/2} + 2(4-0)^{1/2} \right]$$

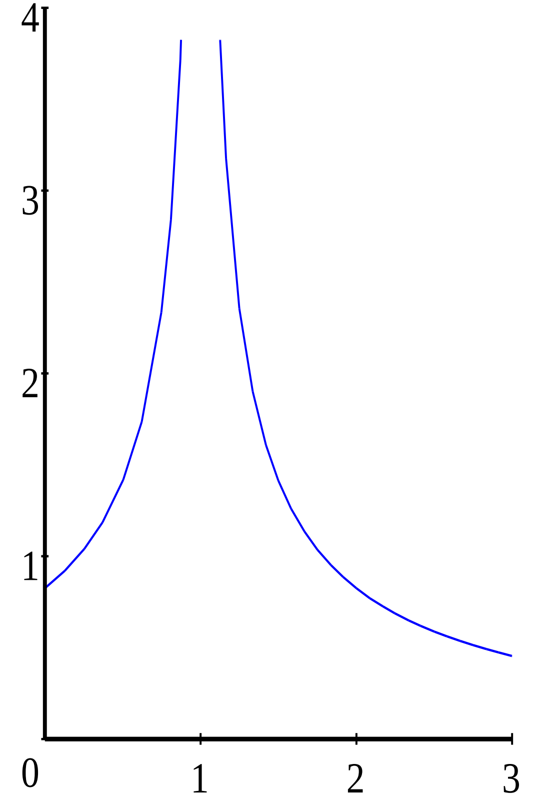
$$0 + 4 = 4$$

Integral converges to 4.

Example 3:

$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

The function approaches ∞ when $x \to 1$.



$$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$$

$$\lim_{c \to 1^-} \int_0^c (x-1)^{-\frac{2}{3}}\ dx + \lim_{c \to 1^+} \int_c^3 (x-1)^{-\frac{2}{3}}\ dx$$

Both integrals must converge for the given integral to converge.

$$\lim_{c \to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^c + \lim_{c \to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_c^3$$

$\rightarrow$

$$\lim_{c\to 1^-}\int_0^c (x-1)^{-\frac{2}{3}}\,dx+\lim_{c\to 1^+}\int_c^3 (x-1)^{-\frac{2}{3}}\,dx$$

$$\lim_{c\to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^c+\lim_{c\to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_c^3$$

$$\lim_{c\to 1^-}\left[\cancel{3(c-1)^{\frac{1}{3}}}^{\,0}-3(-1)^{\frac{1}{3}}\right]+\lim_{c\to 1^+}\left[3\cdot 2^{\frac{1}{3}}-\cancel{3(c-1)^{\frac{1}{3}}}^{\,0}\right]$$

$$(0+3)+(3\sqrt[3]{2}-0)$$

$$\boxed{3+3\sqrt[3]{2}}$$

→

Example 4 $\displaystyle\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2}$

To evaluate this integral, break it into two parts at a convenient value.

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \int_{a}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty} \left[\tan^{-1} x\right]_a^0 = \lim_{a\to-\infty} (\tan^{-1} 0 - \tan^{-1} a)$$

$$= 0 - \frac{-\pi}{2} = \frac{\pi}{2}$$

$$\int_{0}^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int_{0}^{b} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \left[\tan^{-1} x\right]_0^b = \lim_{b\to+\infty} (\tan^{-1} b - \tan^{-1} 0)$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

p. 522 #1, 5, 9, 23, 27, 31, 35, 39, 45.

Improper Integrals continued

Direct Comparison Test

Determine which values of p cause the integral to converges.

$$\int_1^{\infty} \frac{dx}{x^P}$$

Try a negative value for p, p = 0, p = 1, p = 2, etc.

$$\int_1^{\infty} \frac{dx}{x^{-2}} = \int_1^{\infty} x^2 dx = \lim_{a\to\infty} \left[\frac{1}{3}x^3\right]_1^a = \infty - \frac{1}{3} = \infty$$

All negative values of of p diverge

$$\int_1^{\infty} \frac{dx}{x^0} = \int_1^{\infty} 1dx = \lim_{a\to\infty} [x]_1^a = \infty - 1 = \infty$$

P = 0 diverges

$$\int_1^{\infty} \frac{dx}{x^1} = \lim_{a\to\infty} [\ln x]_1^a = \ln\infty - 0 = \infty$$

P = 1 diverges

$$\int_1^{\infty} \frac{dx}{x^2} = \int_1^{\infty} x^{-2} dx = \lim_{a\to\infty} [-1x^{-1}]_1^a = -\frac{1}{\infty} - -1 = 0 + 1 = 1$$

P= 2 converges

$$\int_1^{\infty} \frac{dx}{x^{3/2}} = \int_1^{\infty} x^{-3/2} dx = \lim_{a\to\infty} \left[-2x^{-1/2}\right]_1^a = -\frac{2}{\infty} - -2 = 0 + 2 = 2$$

$$\int_1^\infty \frac{dx}{x^P} \qquad P > 0$$

$$\int_1^\infty x^{-P}\ dx$$

$$\lim_{b\to\infty} \int_1^b x^{-P}\ dx$$

$$\lim_{b\to\infty} \frac{1}{-P+1} x^{-P+1}\Big|_1^b$$

$$\lim_{b\to\infty} \frac{b^{-P+1}}{-P+1} - \frac{1^{-P+1}}{-P+1}$$

If $\mathrm{P} < 1$ then b^{-P+1} gets bigger and bigger as $b \to \infty$, therefore the integral <u>diverges</u>.

If $P > 1$ then b has a negative exponent and $b^{-P+1} \to 0$, therefore the integral <u>converges</u>.

Memorize:

$$\int_1^{\infty} \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

Lower limit does not have to be 1.

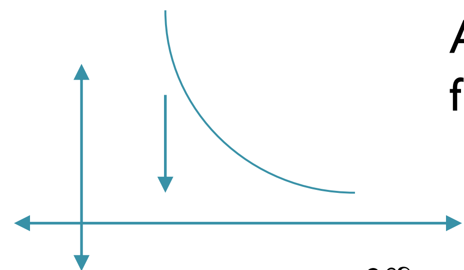
Knowing this "p – integral" is helpful.

When we cannot evaluate an integral directly, we first try to determine whether it converges or diverges by comparing it to known integrals.

Comparison Test

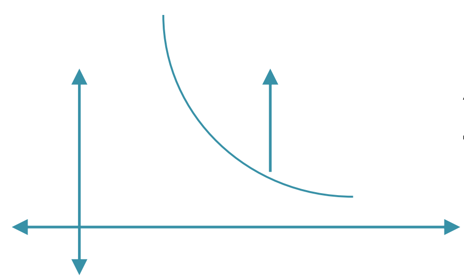
Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.



Anything below a convergent function will also converge.

2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.



Anything above a divergent function will also diverge.

Example 5:

$$\int_1^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

The maximum value of $\sin x = 1$ so:

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty)$$

Since $\dfrac{1}{x^2}$ converges, $\dfrac{\sin^2 x}{x^2}$ converges.

How do we know $1/x^2$ converges?

Because $1/x^2$ is a p-integral with an exponent > 1.

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Example 6:

$$\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\, dx$$

$\sqrt{x^2 - 0.1} < x$ for positive values of x, so:

$$\frac{1}{\sqrt{x^2 - 0.1}} \geq \frac{1}{x} \quad \text{on} \quad [1, \infty)$$

Since $\frac{1}{x}$ diverges, $\frac{1}{\sqrt{x^2 - 0.1}}$ diverges.

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Example 7: $\int_2^{\infty} \frac{1}{x^2 - 1} dx$

$$\frac{1}{x^2 - 1} \geq \frac{1}{x^2}$$

Since the function we are integrating is larger than a convergent function, the Direct Comparison Test is inconclusive.

Homework

Gabriel's Horn paradox worksheet

p.523 #57-62